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On Nirmala Indices of Some Hex-derived Networks of Type Three and Their Subdivision Networks

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Abstract

Several degree-based topological indices have a vital role in the inspection of the chemical properties of various chemical networks. Hex-derived networks, made up of hexagonal mesh networks, have wide applications in the fields of technology, pharmacy and physical sciences. In this research work, we focus on different hex-derived networks of the third type of dimension n and their subdivisions. The Nirmala indices (such as the Nirmala index, the first inverse Nirmala index and the second inverse Nirmala index) are newly introduced degree-based topological indices. Here, we compute the values of these Nirmala indices for the above networks under consideration by operating their standard mathematical formulas and the M-polynomial based method. In addition, we plot the Nirmala indices of the networks and their subdivisions in different dimensions for the purpose of comparative studies among them. The results acquired are helpful in demonstrating the structural properties of considered hex-derived networks and their subdivisions. Also, it may influence the researchers for comparative based studies of the structure and their subdivisions in the sense of the Nirmala indices.

Keywords: Nirmala indices, M-polynomial, Hex-derived networks of third type, Subdivision graph, Graph polynomial.

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1 Introduction

Chemical graph theory (CGT) connects discrete mathematics and chemical structure. A graph G is identified as the ordered pair $(V(G), E(G))$ formed by the vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex v in a graph G is the number of edges that are connected to v and is denoted by $d(v)$ [28]. In the analysis of essential chemistry concepts, mathematical models are crucial. An important element of this model is the topological index. The *topological index* is a mapping from the entire collection of graphs to the set of real numbers that produce the identical value for isomorphic graphs. It is possible to correlate the structure with various physico-chemical properties, biological activities and chemical reactivity using these numerical quantities that correspond to a graph [1, 32].

Early in 2021, V. R. Kulli proposed the Nirmala index [21] of a graph G and represented it as

$$N(G) = \sum_{v_1v_2 \in E(G)} \sqrt{d(v_1) + d(v_2)}.$$

Furthermore, Kulli et al. [22] developed the first and second inverse Nirmala index of a graph in 2021 and these are described as follows.

$$IN_1(G) = \sum_{v_1v_2 \in E(G)} \sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}} = \sum_{v_1v_2 \in E(G)} \left[\frac{1}{d(v_1)} + \frac{1}{d(v_2)}\right]^{\frac{1}{2}},$$

$$IN_2(G) = \sum_{v_1v_2 \in E(G)} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}} = \sum_{v_1v_2 \in E(G)} \left[\frac{1}{d(v_1)} + \frac{1}{d(v_2)}\right]^{-\frac{1}{2}}.$$

Usually, we calculate the topological indices by their formal definitions. Alternatively, we can employ a closed generic approach correlated to polynomials to evaluate the various classes of topological indices. Several polynomials have been proposed in past literature, some of which are the Hosoya polynomial [17], the M-polynomial [15], the NM-polynomial [25], the matching polynomial [16], etc. In 2015, Deutsch and Klavžar proposed the idea of the M-polynomial [15] to calculate the degree-based topological indices by using a single polynomial approach. In the literature, there are several papers [3,4,11–13,18–20,23,26] in which the M-polynomial of different

structures have been calculated. We have determined the M-polynomial formulation of the Nirmala indices in [11] and also verified their correctness by calculating their values for some standard dendrimers.

The algebraic expression of the M-polynomial of a graph G is $M(G;x,y) = \sum_{\delta \le i \le j \le \Delta} m_{i,j}\ x^i y^j$, where $\delta = \min\{d(v)|v \in V(G)\}$, $\Delta = \max\{d(v)|v \in V(G)\}$ and $m_{i,j}$ is the number of edges $v_1v_2 \in E(G)$ such that $d(v_1) = i$, $d(v_2) = j$ $(i,j \ge 1)$. In [14], it is described that we can express any degree-based topological indices for a graph G as $I(G) = \sum_{v_1v_2 \in E(G)} f(d(v_1), d(v_2)) = \sum_{i \le j} m_{i,j} f(i,j)$, where $f(d(v_1), d(v_2))$ is the function of $d(v_1)$ and $d(v_2)$ and for the Nirmala indices it is listed in Table 1.

Table 1: Derivation formulas of Nirmala indices from M-polynomial of a graph G.

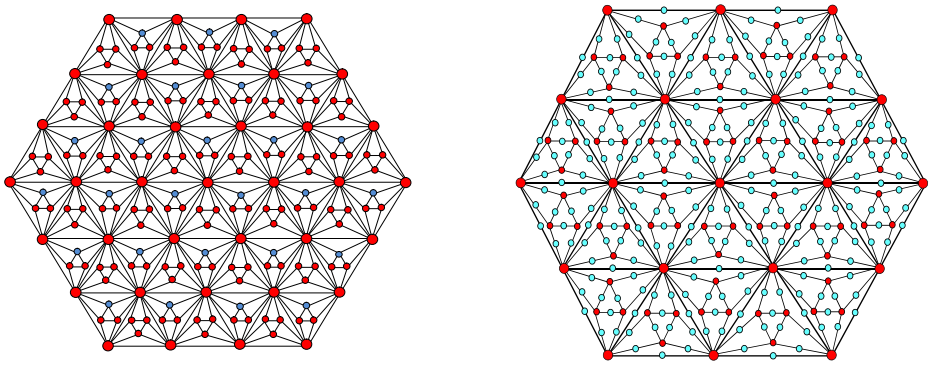
Sl. No.	**Topological Index**	**Notation**	$f(x,y)$	**Derivation from** $M(G;x,y)$
1.	Nirmala Index	$N(G)$	$\sqrt{x+y}$	$D_x^{1/2} J M(G;x,y)\|_{x=1}$
2.	First Inverse Nirmala Index	$IN_1(G)$	$\sqrt{\frac{1}{x}+\frac{1}{y}}$	$D_x^{1/2} J S_y^{1/2} S_x^{1/2} M(G;x,y)\|_{x=1}$
3.	Second Inverse Nirmala Index	$IN_2(G)$	$\frac{1}{\sqrt{\frac{1}{x}+\frac{1}{y}}}$	$S_x^{1/2} J D_y^{1/2} D_x^{1/2} M(G;x,y)\|_{x=1}$
Where, $D_x^{1/2}(f(x,y)) = \sqrt{x\frac{\partial(f(x,y))}{\partial x}} \cdot \sqrt{f(x,y)}$, $D_y^{1/2}(f(x,y)) = \sqrt{y\frac{\partial(f(x,y))}{\partial y}} \cdot \sqrt{f(x,y)}$, $S_x^{1/2}(f(x,y)) = \sqrt{\int_0^x \frac{f(t,y)}{t}dt} \cdot \sqrt{f(x,y)}$, $S_y^{1/2}(f(x,y)) = \sqrt{\int_0^y \frac{f(x,t)}{t}dt} \cdot \sqrt{f(x,y)}$, $J(f(x,y)) = f(x,x)$.				

This table also lists the derivation formulas of the Nirmala indices from the M-polynomial of a graph G [11].

2 Different Hex-derived Networks and Their Subdivisions

In 1990, Chen et al. [2] developed a processor's addressing method as well as associated broadcasting and routing algorithms for interconnected hexagonal networks. Planar graphs are used to model hexagonal networks. Triangular plane tessellation, or the division of a plane into equilateral triangles, is the basis of these networks. The honeycomb networks and mesh networks are the closest networks in which the honeycomb networks are built on regular hexagons, whereas mesh networks are based on a regular square partition. They are used in wireless and connectivity networks, computer graphics and in image processing to design benzenoid hydrocarbons in chemistry [27]. In 2008, the hexagonal network has been used to create two

types of hex-derived networks, they are the hex-derived network of type one of dimension n ($HDN1[n]$) and the hex-derived network of type two of dimension n ($HDN2[n]$) [24]. Aside from that, new chemical networks, named the third type of hex-derived networks [31], have been constructed by Raj and George in 2017. They are the hex-derived network of third type of dimension n (represented as $HDN3[n]$, see Figure 1(a) for $HDN3[4]$), the triangular hex-derived network of third type of dimension n (represented as $THDN3[n]$, see Figure 2(a) for $THDN3[7]$), the rectangular hex-derived network of third type of dimension n (represented as $RHDN3[n]$, see Figure 3(a) for $RHDN3[4]$) and the chain hex-derived network of third type of dimension n (represented as $CHDN3[n]$), see Figure 4(a) for $CHDN3[5]$). For more details, please refer to [5–7, 9] and their references therein.



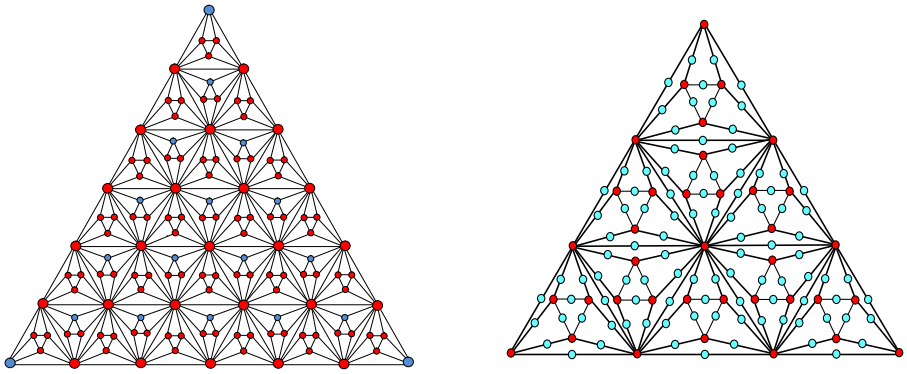
(a) Hex-derived network of third type ($HDN3[4]$).

(b) Subdivided hex-derived network of third type ($SHDN3[3]$).

Figure 1: Examples of a structure of the hex-derived and subdivided hex-derived networks.

By performing graph operations such as subdivision, contraction, removal, or inclusion of an edge or a vertex, or merger, we can frame a new graph from an existing one. A graph G is subdivided by inserting a new vertex between the edge. Using this fundamental subdivision procedure on the $HDN3[n]$, $THDN3[n]$, $RHDN3[n]$ and $CHDN3[n]$ networks, one can have a subdivided hex-derived network of third type of dimension n (symbolized as $SHDN3[n]$, see Figure 1(b) for $SHDN3[3]$), subdivided triangular hex-derived network of third type of dimension n (symbolized as $STHDN3[n]$, see Figure 2(b) for $STHDN3[4]$), subdivided rectangular hex-derived network of third type of dimension n (symbolized as $SRHDN3[n]$, see Figure 3(b)

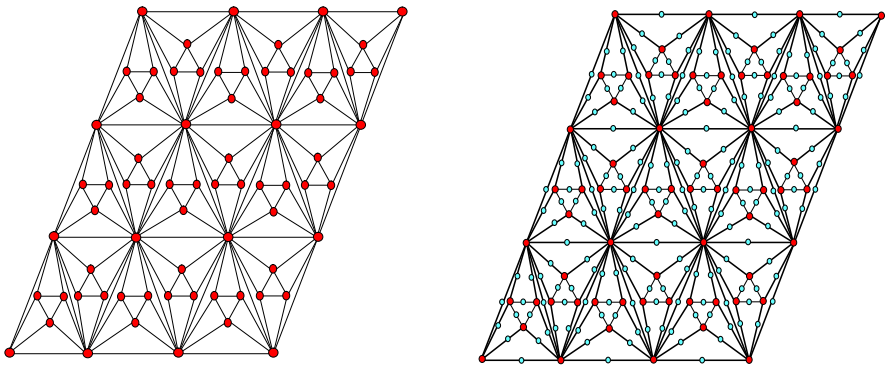
for $SRHDN3[4]$) and subdivided chain hex-derived network of third type of dimension n (symbolized as $SCHDN3[n]$, see Figure 4(b) for $SCHDN3[5]$). Please see [8, 10, 29, 30] and their references therein for more detailed structural information.



(a) Triangular hex-derived network of third type ($THDN3[7]$).

(b) Subdivided triangular hex-derived network of third type ($STHDN3[4]$).

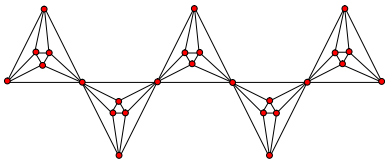
Figure 2: Examples of a structure of the triangular and subdivided triangular hex-derived networks.



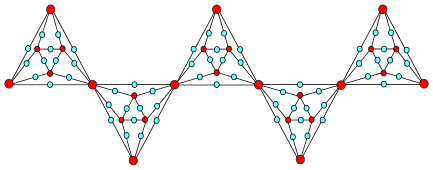
(a) Rectangular hex-derived network of third type ($RHDN3[4]$).

(b) Subdivided rectangular hex-derived network of third type ($SRHDN3[4]$).

Figure 3: Examples of a structure of the rectangular and subdivided rectangular hex-derived networks.



(a) Chain hex-derived network of third type ($CHDN3[5]$).



(b) Subdivided chain hex-derived network of third type ($SCHDN3[5]$).

Figure 4: Examples of a structure of the chain and subdivided chain hex-derived networks.

3 Our Perspective and Direction

In this research article, we have taken the hex-derived networks of type three of dimension n along with their subdivisions, and examples of their structures are delineated in Figures 1 to 4, respectively. We aimed to evaluate the Nirmala indices for these hex-derived networks and their subdivisions by applying two approaches. The first approach is operating the direct mathematical formulas of the Nirmala indices, and the second by applying their M-polynomial based formulas. In Section 4.1, we evaluate the Nirmala indices for the networks using the standard mathematical degree-based formula approach. Subsequently, by employing the M-polynomial based derivation formulas of the Nirmala indices, we calculate their values for our considered network in Section 4.2. Moreover, to analyze the behavior of the Nirmala indices and relationships among them for different hex-derived networks and their allied subdivisions, we plot them in Section 5, considering the various perspectives. At last, in Section 6, we conclude with some notable remarks.

4 Determining the Nirmala Indices for Different Hex-derived Networks of Type Three and Their Subdivisions

This section specifies the determination of the Nirmala indices for the hex-derived networks of type three of dimension n and their subdivisions using their degree-dependent mathematical formulas and the M-polynomial based formulas.

4.1 Computing the Nirmala Indices Using the Degree-Dependent Mathematical Formula Approach

In this section, we directly operate the standard mathematical formulas of the Nirmala indices (as discussed in Section 1) to determine their values for the hex-derived networks and corresponding subdivided hex-derived networks.

Theorem 4.1 *Let HDN*3[n] *represents the hex-derived network of third type of dimension* n*. Then the Nirmala indices for the HDN*3[n] *network, where* $n \geq 4$*, is given by*

(i) $N(HDN3[n]) = (36\sqrt{2}+36\sqrt{22}+54)n^2+(-72\sqrt{2}+36\sqrt{14}-108\sqrt{22}+12\sqrt{5}+24\sqrt{7}-198)n+36\sqrt{2}+24\sqrt{11}-72\sqrt{14}+84\sqrt{22}+12\sqrt{17}+210-36\sqrt{5}-48\sqrt{7}$,

(ii) $IN_1(HDN3[n]) = (9\sqrt{2}+6\sqrt{11}+3)n^2+(-18\sqrt{2}+\frac{22\sqrt{35}}{5}-18\sqrt{11}+\frac{6\sqrt{5}}{5}-11)n+9\sqrt{2}+\frac{12\sqrt{77}}{7}-\frac{44\sqrt{35}}{5}+14\sqrt{11}+\frac{6\sqrt{1190}}{35}+\frac{5\sqrt{14}}{7}-\frac{18\sqrt{5}}{5}+10$,

(iii) $IN_2(HDN3[n]) = (18\sqrt{2}+\frac{216\sqrt{11}}{11}+27)n^2+(-36\sqrt{2}+\frac{72\sqrt{35}}{7}-\frac{648\sqrt{11}}{11}+6\sqrt{5}+\frac{36\sqrt{35}}{7}-99)n+18\sqrt{2}+\frac{48\sqrt{77}}{11}-\frac{144\sqrt{35}}{7}+\frac{504\sqrt{11}}{11}+\frac{12\sqrt{1190}}{17}+\frac{18\sqrt{14}}{5}-18\sqrt{5}-\frac{72\sqrt{35}}{7}+90$.

Proof: Table 2 lists the edge set partition of the *HDN*3[n] network based on the degree of end vertices of each edge [5]. These values are required for determining the Nirmala indices of the network by using their degree-dependent formulas (discussed in Section 1), as shown below.

Table 2: The edge set partition of the *HDN*3[n] network based on the degree of end vertices of each edge.

Distinct pair of edges	$(4,4)$	$(4,7)$	$(4,10)$	$(4,18)$	$(7,10)$	$(7,18)$	$(10,10)$	$(10,18)$	$(18,18)$
Number of such pairs	$18n^2-36n+18$	24	$36n-72$	$36n^2-108n+84$	12	6	$6n-18$	$12n-24$	$9n^2-33n+30$

(i) **Nirmala Index:**

$$N(HDN3[n]) = \sum_{v_1v_2 \in E(HDN3[n])} \sqrt{d(v_1)+d(v_2)}$$

$$
\begin{aligned}
&= (18n^2 - 36n + 18)\sqrt{4+4} + 24\sqrt{4+7} + (36n-72)\sqrt{4+10} \\
&\quad + (36n^2 - 108n + 84)\sqrt{4+18} + 12\sqrt{7+10} + 6\sqrt{7+18} \\
&\quad + (6n-18)\sqrt{10+10} + (12n-24)\sqrt{10+18} \\
&\quad + (9n^2 - 33n + 30)\sqrt{18+18} \\
&= 2\sqrt{2}(18n^2 - 36n + 18) + 24\sqrt{11} + \sqrt{14}(36n-72) \\
&\quad + \sqrt{22}(36n^2 - 108n + 84) + 12\sqrt{17} + 30 + 2\sqrt{5}(6n-18) \\
&\quad + 2\sqrt{7}(12n-24) + 6(9n^2 - 33n + 30) \\
&= (36\sqrt{2} + 36\sqrt{22} + 54)n^2 \\
&\quad + (-72\sqrt{2} + 36\sqrt{14} - 108\sqrt{22} + 12\sqrt{5} + 24\sqrt{7} - 198)n \\
&\quad + 36\sqrt{2} + 24\sqrt{11} - 72\sqrt{14} + 84\sqrt{22} + 12\sqrt{17} + 210 - 36\sqrt{5} - 48\sqrt{7}.
\end{aligned}
$$

(*ii*) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(HDN3[n]) \\
&= \sum_{v_1v_2 \in E(HDN3[n])} \sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}} \\
&= (18n^2 - 36n + 18)\sqrt{\frac{1}{4} + \frac{1}{4}} + 24\sqrt{\frac{1}{4} + \frac{1}{7}} + (36n-72)\sqrt{\frac{1}{4} + \frac{1}{10}} \\
&\quad + (36n^2 - 108n + 84)\sqrt{\frac{1}{4} + \frac{1}{18}} + 12\sqrt{\frac{1}{7} + \frac{1}{10}} + 6\sqrt{\frac{1}{7} + \frac{1}{18}} \\
&\quad + (6n-18)\sqrt{\frac{1}{10} + \frac{1}{10}} + (12n-24)\sqrt{\frac{1}{10} + \frac{1}{18}} \\
&\quad + (9n^2 - 33n + 30)\sqrt{\frac{1}{18} + \frac{1}{18}} \\
&= \frac{\sqrt{2}}{2}(18n^2 - 36n + 18) + \frac{12}{7}\sqrt{77} + \frac{\sqrt{35}}{10}(36n-72) \\
&\quad + \frac{\sqrt{11}}{6}(36n^2 - 108n + 84) + \frac{6}{35}\sqrt{1190} + \frac{5}{7}\sqrt{14} + \frac{\sqrt{5}}{5}(6n-18) \\
&\quad + \frac{\sqrt{35}}{15}(12n-24) + (3n^2 - 11n + 10) \\
&= (9\sqrt{2} + 6\sqrt{11} + 3)n^2 + (-18\sqrt{2} + \frac{22\sqrt{35}}{5} - 18\sqrt{11} + \frac{6\sqrt{5}}{5} - 11)n \\
&\quad + 9\sqrt{2} + \frac{12\sqrt{77}}{7} - \frac{44\sqrt{35}}{5} + 14\sqrt{11} + \frac{6\sqrt{1190}}{35} + \frac{5\sqrt{14}}{7} - \frac{18\sqrt{5}}{5} + 10.
\end{aligned}
$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(HDN3[n])\\
&= \sum_{v_1v_2\in E(HDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)}+\frac{1}{d(v_2)}}}\\
&= (18n^2-36n+18)\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}} + 24\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{7}}} + (36n-72)\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{10}}}\\
&\quad + (36n^2-108n+84)\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{18}}} + 12\frac{1}{\sqrt{\frac{1}{7}+\frac{1}{10}}} + 6\frac{1}{\sqrt{\frac{1}{7}+\frac{1}{18}}}\\
&\quad + (6n-18)\frac{1}{\sqrt{\frac{1}{10}+\frac{1}{10}}} + (12n-24)\frac{1}{\sqrt{\frac{1}{10}+\frac{1}{18}}}\\
&\quad + (9n^2-33n+30)\frac{1}{\sqrt{\frac{1}{18}+\frac{1}{18}}}\\
&= \sqrt{2}(18n^2-36n+18) + \frac{48}{11}\sqrt{77} + \frac{\sqrt{140}}{7}(36n-72)\\
&\quad + \frac{6\sqrt{11}}{11}(36n^2-108n+84) + \frac{12}{17}\sqrt{1190} + \frac{18}{5}\sqrt{14}\\
&\quad + \sqrt{5}(6n-18) + \frac{3\sqrt{35}}{7}(12n-24) + 3(9n^2-33n+30)\\
&= (18\sqrt{2}+\frac{216\sqrt{11}}{11}+27)n^2 + (-36\sqrt{2}+\frac{72\sqrt{35}}{7}-\frac{648\sqrt{11}}{11}+6\sqrt{5}\\
&\quad + \frac{36\sqrt{35}}{7}-99)n + 18\sqrt{2} + \frac{48\sqrt{77}}{11} - \frac{144\sqrt{35}}{7} + \frac{504\sqrt{11}}{11}\\
&\quad + \frac{12\sqrt{1190}}{17} + \frac{18\sqrt{14}}{5} - 18\sqrt{5} - \frac{72\sqrt{35}}{7} + 90.
\end{aligned}
$$

□

Theorem 4.2 *Let SHDN*3[n] *represents the subdivided hex-derived network of third type of dimension* n*. Then the Nirmala indices for the SHDN*3[n] *network, where* $n \geq 3$*, is given by*

(i) $N(SHDN3[n]) = (72\sqrt{6}+108\sqrt{5})n^2 + (-144\sqrt{6}+120\sqrt{3}-324\sqrt{5})n + 72\sqrt{6}+126-240\sqrt{3}+252\sqrt{5}$,

(ii) $IN_1(SHDN3[n]) = (36\sqrt{3}+18\sqrt{5})n^2 + (-72\sqrt{3}+12\sqrt{15}-54\sqrt{5})n + 36\sqrt{3}+9\sqrt{14}-24\sqrt{15}+42\sqrt{5}$,

(iii) $IN_2(SHDN3[n]) = (48\sqrt{3} + \frac{162\sqrt{5}}{5})n^2 + (-96\sqrt{3} + 20\sqrt{15} - \frac{486\sqrt{5}}{5})n + 48\sqrt{3} + 14\sqrt{14} - 40\sqrt{15} + \frac{378\sqrt{5}}{5}$.

Proof: Using the degree-dependent formulas and the edge set partition (see Table 3) of the $SHDN3[n]$ network, the Nirmala indices of the network are calculated below.

Table 3: The edge set partition of the $SHDN3[n]$ network based on the degree of end vertices of each edge [10].

Distinct pair of edges	$(2,4)$	$(2,7)$	$(2,10)$	$(2,18)$
Number of such pairs	$72(n-1)^2$	42	$60(n-2)$	$18(3n^2-9n+7)$

(i) **Nirmala Index:**

$$
\begin{aligned}
&N(SHDN3[n]) \\
&= \sum_{v_1v_2 \in E(SHDN3[n])} \sqrt{d(v_1) + d(v_2)} \\
&= 72(n-1)^2\sqrt{2+4} + 42\sqrt{2+7} + 60(n-2)\sqrt{2+10} + 18(3n^2 - 9n \\
&\quad + 7)\sqrt{2+18} \\
&= 72\sqrt{6}(n-1)^2 + 42\sqrt{9} + 120\sqrt{3}(n-2) + 36\sqrt{5}(3n^2 - 9n + 7) \\
&= (72\sqrt{6} + 108\sqrt{5})n^2 + (-144\sqrt{6} + 120\sqrt{3} - 324\sqrt{5})n + 72\sqrt{6} + 126 \\
&\quad - 240\sqrt{3} + 252\sqrt{5}.
\end{aligned}
$$

(ii) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(SHDN3[n]) \\
&= \sum_{v_1v_2 \in E(SHDN3[n])} \sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}} \\
&= 72(n-1)^2\sqrt{\frac{1}{2}+\frac{1}{4}} + 42\sqrt{\frac{1}{2}+\frac{1}{7}} + 60(n-2)\sqrt{\frac{1}{2}+\frac{1}{10}} \\
&\quad + 18(3n^2 - 9n + 7)\sqrt{\frac{1}{2}+\frac{1}{18}} \\
&= 36\sqrt{3}(n-1)^2 + 9\sqrt{14} + 12\sqrt{15}(n-2) + 6\sqrt{5}(3n^2 - 9n + 7)
\end{aligned}
$$

$$= (36\sqrt{3} + 18\sqrt{5})n^2 + (-72\sqrt{3} + 12\sqrt{15} - 54\sqrt{5})n + 36\sqrt{3} + 9\sqrt{14} - 24\sqrt{15} + 42\sqrt{5}.$$

(*iii*) **Second Inverse Nirmala Index:**

$$\begin{aligned}
&IN_2(SHDN3[n]) \\
&= \sum_{v_1v_2 \in E(SHDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}} \\
&= 72(n-1)^2 \frac{1}{\sqrt{\frac{1}{2} + \frac{1}{4}}} + 42 \frac{1}{\sqrt{\frac{1}{2} + \frac{1}{7}}} + 60(n-2) \frac{1}{\sqrt{\frac{1}{2} + \frac{1}{10}}} \\
&\quad + 18(3n^2 - 9n + 7) \frac{1}{\sqrt{\frac{1}{2} + \frac{1}{18}}} \\
&= 48\sqrt{3}(n-1)^2 + 14\sqrt{14} + 20\sqrt{15}(n-2) + \frac{54\sqrt{5}}{5}(3n^2 - 9n + 7) \\
&= (48\sqrt{3} + \frac{162\sqrt{5}}{5})n^2 + (-96\sqrt{3} + 20\sqrt{15} - \frac{486\sqrt{5}}{5})n + 48\sqrt{3} \\
&\quad + 14\sqrt{14} - 40\sqrt{15} + \frac{378\sqrt{5}}{5}.
\end{aligned}$$

□

Theorem 4.3 *Let THDN*3[n] *represents the triangular hex-derived network of third type of dimension* n*. Then the Nirmala indices for the THDN*3[n] *network, where* $n \geq 4$*, is given by*

(i) $N(THDN3[n]) = (6\sqrt{2} + 6\sqrt{22} + 9)n^2 + (-12\sqrt{2} + 18\sqrt{14} - 30\sqrt{22} + 6\sqrt{5} + 12\sqrt{7} - 63)n + 18\sqrt{2} - 30\sqrt{14} + 36\sqrt{22} - 12\sqrt{5} - 36\sqrt{7} + 108,$

(ii) $IN_1(THDN3[n]) = (\frac{3\sqrt{2}}{2} + \sqrt{11} + \frac{1}{2})n^2 + (-3\sqrt{2} + \frac{11\sqrt{35}}{5} - 5\sqrt{11} + \frac{3\sqrt{5}}{5} - \frac{7}{2})n + \frac{9\sqrt{2}}{2} - \frac{21\sqrt{35}}{5} + 6\sqrt{11} - \frac{6\sqrt{5}}{5} + 6,$

(iii) $IN_2(THDN3[n]) = (3\sqrt{2} + \frac{36\sqrt{11}}{11} + \frac{9}{2})n^2 + (-6\sqrt{2} + \frac{36\sqrt{35}}{7} - \frac{180\sqrt{11}}{11} + 3\sqrt{5} + \frac{18\sqrt{35}}{7} - \frac{63}{2})n + 9\sqrt{2} - \frac{60\sqrt{35}}{7} + \frac{216\sqrt{11}}{11} - 6\sqrt{5} - \frac{54\sqrt{35}}{7} + 54.$

Proof: Based on the degree of the end vertices of each edge, Table 4 displays the edge set partition of the *THDN*3[n] network. These data are essential for computing the Nirmala indices of the network by using their degree-dependent formulas, as demonstrated below.

Table 4: The edge set partition of the $THDN3[n]$ network based on the degree of end vertices of each edge [7].

Distinct pair of edges	$(4,4)$	$(4,10)$	$(4,18)$	$(10,10)$	$(10,18)$	$(18,18)$
Number of such pairs	$3n^2-6n+9$	$18n-30$	$6n^2-30n+36$	$3n-6$	$6n-18$	$\frac{1}{2}(3n^2-21n+36)$

(i) **Nirmala Index:**

$$\begin{aligned}
&N(THDN3[n])\\
&=\sum_{v_1v_2\in E(THDN3[n])}\sqrt{d(v_1)+d(v_2)}\\
&=(3n^2-6n+9)\sqrt{4+4}+(18n-30)\sqrt{4+10}\\
&\quad+(6n^2-30n+36)\sqrt{4+18}+(3n-6)\sqrt{10+10}\\
&\quad+(6n-18)\sqrt{10+18}+\frac{1}{2}(3n^2-21n+36)\sqrt{18+18}\\
&=2\sqrt{2}(3n^2-6n+9)+\sqrt{14}(18n-30)+\sqrt{22}(6n^2-30n+36)\\
&\quad+2\sqrt{5}(3n-6)+2\sqrt{7}(6n-18)+3(3n^2-21n+36)\\
&=(6\sqrt{2}+6\sqrt{22}+9)n^2+(-12\sqrt{2}+18\sqrt{14}-30\sqrt{22}+6\sqrt{5}+12\sqrt{7}\\
&\quad-63)n+18\sqrt{2}-30\sqrt{14}+36\sqrt{22}-12\sqrt{5}-36\sqrt{7}+108.
\end{aligned}$$

(ii) **First Inverse Nirmala Index:**

$$\begin{aligned}
&IN_1(THDN3[n])\\
&=\sum_{v_1v_2\in E(THDN3[n])}\sqrt{\frac{1}{d(v_1)}+\frac{1}{d(v_2)}}\\
&=(3n^2-6n+9)\sqrt{\frac{1}{4}+\frac{1}{4}}+(18n-30)\sqrt{\frac{1}{4}+\frac{1}{10}}\\
&\quad+(6n^2-30n+36)\sqrt{\frac{1}{4}+\frac{1}{18}}+(3n-6)\sqrt{\frac{1}{10}+\frac{1}{10}}\\
&\quad+(6n-18)\sqrt{\frac{1}{10}+\frac{1}{18}}+\frac{1}{2}(3n^2-21n+36)\sqrt{\frac{1}{18}+\frac{1}{18}}\\
&=\frac{\sqrt{2}}{2}(3n^2-6n+9)+\frac{\sqrt{35}}{10}(18n-30)+\frac{\sqrt{11}}{6}(6n^2-30n+36)\\
&\quad+\frac{\sqrt{5}}{5}(3n-6)+\frac{\sqrt{35}}{15}(6n-18)+\frac{1}{6}(3n^2-21n+36)
\end{aligned}$$

$$= (\frac{3\sqrt{2}}{2} + \sqrt{11} + \frac{1}{2})n^2 + (-3\sqrt{2} + \frac{11\sqrt{35}}{5} - 5\sqrt{11} + \frac{3\sqrt{5}}{5} - \frac{7}{2})n + \frac{9\sqrt{2}}{2} - \frac{21\sqrt{35}}{5} + 6\sqrt{11} - \frac{6\sqrt{5}}{5} + 6.$$

(*iii*) **Second Inverse Nirmala Index:**

$$\begin{aligned}
&IN_2(THDN3[n]) \\
&= \sum_{v_1v_2 \in E(THDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}} \\
&= (3n^2 - 6n + 9)\frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} + (18n - 30)\frac{1}{\sqrt{\frac{1}{4} + \frac{1}{10}}} \\
&\quad + (6n^2 - 30n + 36)\frac{1}{\sqrt{\frac{1}{4} + \frac{1}{18}}} + (3n - 6)\frac{1}{\sqrt{\frac{1}{10} + \frac{1}{10}}} \\
&\quad + (6n - 18)\frac{1}{\sqrt{\frac{1}{10} + \frac{1}{18}}} + \frac{1}{2}(3n^2 - 21n + 36)\frac{1}{\sqrt{\frac{1}{18} + \frac{1}{18}}} \\
&= \sqrt{2}(3n^2 - 6n + 9) + \frac{2\sqrt{35}}{7}(18n - 30) + \frac{6\sqrt{11}}{11}(6n^2 - 30n + 36) \\
&\quad + \sqrt{5}(3n - 6) + \frac{3\sqrt{35}}{7}(6n - 18) + \frac{3}{2}(3n^2 - 21n + 36) \\
&= (3\sqrt{2} + \frac{36\sqrt{11}}{11} + \frac{9}{2})n^2 + (-6\sqrt{2} + \frac{36\sqrt{35}}{7} - \frac{180\sqrt{11}}{11} + 3\sqrt{5} \\
&\quad + \frac{18\sqrt{35}}{7} - \frac{63}{2})n + 9\sqrt{2} - \frac{60\sqrt{35}}{7} + \frac{216\sqrt{11}}{11} - 6\sqrt{5} - \frac{54\sqrt{35}}{7} + 54.
\end{aligned}$$

□

Theorem 4.4 *Let STHDN*3[n] *represents the subdivided triangular hex-derived network of third type of dimension* n*. Then the Nirmala indices for the STHDN*3[n] *network, where* $n \geq 4$*, is given by*

(i) $N(STHDN3[n]) = (12\sqrt{6} + 18\sqrt{5})n^2 + (-24\sqrt{6} + 60\sqrt{3} - 90\sqrt{5})n + 24\sqrt{6} - 120\sqrt{3} + 108\sqrt{5}$,

(ii) $IN_1(STHDN3[n]) = (6\sqrt{3} + 3\sqrt{5})n^2 + (-12\sqrt{3} + 6\sqrt{15} - 15\sqrt{5})n + 12\sqrt{3} - 12\sqrt{15} + 18\sqrt{5}$,

(iii) $IN_2(STHDN3[n]) = (8\sqrt{3} + \frac{27\sqrt{5}}{5})n^2 + (-16\sqrt{3} + 10\sqrt{15} - 27\sqrt{5})n + 16\sqrt{3} - 20\sqrt{15} + \frac{162\sqrt{5}}{5}$.

Proof: Below, we determine the Nirmala indices of the $STHDN3[n]$ network by using their degree-dependent formulas, and the edge set partition (see Table 5) of the network.

Table 5: The edge set partition of the $STHDN3[n]$ network based on the degree of end vertices of each edge [30].

Distinct pair of edges	$(2,4)$	$(2,10)$	$(2,18)$
Number of such pairs	$12(n^2-2n+2)$	$30(n-2)$	$9(n-2)(n-3)$

(*i*) **Nirmala Index:**

$$\begin{aligned}
&N(STHDN3[n])\\
&= \sum_{v_1v_2 \in E(STHDN3[n])} \sqrt{d(v_1) + d(v_2)}\\
&= 12(n^2-2n+2)\sqrt{2+4} + 30(n-2)\sqrt{2+10}\\
&\quad + 9(n-2)(n-3)\sqrt{2+18}\\
&= 12\sqrt{6}(n^2-2n+2) + 60\sqrt{3}(n-2) + 18\sqrt{5}(n-2)(n-3)\\
&= (12\sqrt{6} + 18\sqrt{5})n^2 + (-24\sqrt{6} + 60\sqrt{3} - 90\sqrt{5})n\\
&\quad + 24\sqrt{6} - 120\sqrt{3} + 108\sqrt{5}.
\end{aligned}$$

(*ii*) **First Inverse Nirmala Index:**

$$\begin{aligned}
&IN_1(STHDN3[n])\\
&= \sum_{v_1v_2 \in E(STHDN3[n])} \sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}\\
&= 12(n^2-2n+2)\sqrt{\frac{1}{2}+\frac{1}{4}} + 30(n-2)\sqrt{\frac{1}{2}+\frac{1}{10}}\\
&\quad + 9(n-2)(n-3)\sqrt{\frac{1}{2}+\frac{1}{18}}\\
&= 6\sqrt{3}(n^2-2n+2) + 6\sqrt{15}(n-2) + 3\sqrt{5}(n-2)(n-3)
\end{aligned}$$

$$= (6\sqrt{3} + 3\sqrt{5})n^2 + (-12\sqrt{3} + 6\sqrt{15} - 15\sqrt{5})n + 12\sqrt{3} - 12\sqrt{15} + 18\sqrt{5}.$$

(*iii*) **Second Inverse Nirmala Index:**

$$\begin{aligned}
&IN_2(STHDN3[n]) \\
&= \sum_{v_1v_2 \in E(STHDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}} \\
&= 12(n^2 - 2n + 2)\frac{1}{\sqrt{\frac{1}{2} + \frac{1}{4}}} + 30(n-2)\frac{1}{\sqrt{\frac{1}{2} + \frac{1}{10}}} \\
&\quad + 9(n-2)(n-3)\frac{1}{\sqrt{\frac{1}{2} + \frac{1}{18}}} \\
&= 8\sqrt{3}(n^2 - 2n + 2) + 10\sqrt{15}(n-2) + \frac{27\sqrt{5}}{5}(n-2)(n-3) \\
&= (8\sqrt{3} + \frac{27\sqrt{5}}{5})n^2 + (-16\sqrt{3} + 10\sqrt{15} - 27\sqrt{5})n \\
&\quad + 16\sqrt{3} - 20\sqrt{15} + \frac{162\sqrt{5}}{5}.
\end{aligned}$$

□

Theorem 4.5 *Let RHDN*3[n] *represents the rectangular hex-derived network of third type of dimension* n*. Then the Nirmala indices for the RHDN*3[n] *network, where* $n \geq 4$*, is given by*

(i) $N(RHDN3[n]) = (12\sqrt{2}+12\sqrt{22}+18)n^2+(-24\sqrt{2}+24\sqrt{14}-48\sqrt{22}+8\sqrt{5}+16\sqrt{7}-96)n+20\sqrt{2}+8\sqrt{11}-44\sqrt{14}+48\sqrt{22}+4\sqrt{17}+136-20\sqrt{5}-40\sqrt{7}$,

(ii) $IN_1(RHDN3[n]) = (3\sqrt{2}+2\sqrt{11}+1)n^2+(-6\sqrt{2}+\frac{44\sqrt{35}}{15}-8\sqrt{11}+\frac{4\sqrt{5}}{5}-\frac{16}{3})n+5\sqrt{2}+\frac{4\sqrt{77}}{7}-\frac{86\sqrt{35}}{15}+8\sqrt{11}+\frac{2\sqrt{1190}}{35}+\frac{5\sqrt{14}}{21}-2\sqrt{5}+7$,

(iii) $IN_2(RHDN3[n]) = (6\sqrt{2}+\frac{72\sqrt{11}}{11}+9)n^2+(-12\sqrt{2}+\frac{48\sqrt{35}}{7}-\frac{288\sqrt{11}}{11}+4\sqrt{5}+\frac{24\sqrt{35}}{7}-48)n+10\sqrt{2}+\frac{16\sqrt{77}}{11}-\frac{88\sqrt{35}}{7}+\frac{288\sqrt{11}}{11}+\frac{4\sqrt{1190}}{17}+\frac{6\sqrt{14}}{5}-10\sqrt{5}-\frac{60\sqrt{35}}{7}+63$.

Proof: Based on the degree of the end vertices of each edge, Table 6 shows the edge set partition of the $RHDN3[n]$ network that is necessary for calculating the Nirmala indices of the network by using their degree-dependent formulas, as shown below.

Table 6: The edge set partition of the $RHDN3[n]$ network based on the degree of end vertices of each edge [9].

Distinct pair of edges	$(4,4)$	$(4,7)$	$(4,10)$	$(4,18)$	$(7,10)$	$(7,18)$	$(10,10)$	$(10,18)$	$(18,18)$
Number of such pairs	$6n^2-12n+10$	8	$24n-44$	$12n^2-48n+48$	4	2	$4n-10$	$8n-20$	$3n^2-16n+21$

(i) **Nirmala Index:**

$$
\begin{aligned}
&N(RHDN3[n])\\
&= \sum_{v_1v_2\in E(RHDN3[n])} \sqrt{d(v_1)+d(v_2)}\\
&= (6n^2-12n+10)\sqrt{4+4}+8\sqrt{4+7}+(24n-44)\sqrt{4+10}\\
&\quad+(12n^2-48n+48)\sqrt{4+18}+4\sqrt{7+10}+2\sqrt{7+18}\\
&\quad+(4n-10)\sqrt{10+10}+(8n-20)\sqrt{10+18}\\
&\quad+(3n^2-16n+21)\sqrt{18+18}\\
&= 2\sqrt{2}(6n^2-12n+10)+8\sqrt{11}+\sqrt{14}(24n-44)\\
&\quad+\sqrt{22}(12n^2-48n+48)+4\sqrt{17}+10+2\sqrt{5}(4n-10)\\
&\quad+2\sqrt{7}(8n-20)+6(3n^2-16+21)\\
&= (12\sqrt{2}+12\sqrt{22}+18)n^2+(-24\sqrt{2}+24\sqrt{14}-48\sqrt{22}+8\sqrt{5}\\
&\quad+16\sqrt{7}-96)n+20\sqrt{2}+8\sqrt{11}-44\sqrt{14}+48\sqrt{22}+4\sqrt{17}+136\\
&\quad-20\sqrt{5}-40\sqrt{7}.
\end{aligned}
$$

(ii) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(RHDN3[n])\\
&= \sum_{v_1v_2\in E(RHDN3[n])} \sqrt{\frac{1}{d(v_1)}+\frac{1}{d(v_2)}}\\
&= (6n^2-12n+10)\sqrt{\frac{1}{4}+\frac{1}{4}}+8\sqrt{\frac{1}{4}+\frac{1}{7}}+(24n-44)\sqrt{\frac{1}{4}+\frac{1}{10}}
\end{aligned}
$$

$$
\begin{aligned}
&+ (12n^2 - 48n + 48)\sqrt{\frac{1}{4} + \frac{1}{18}} + 4\sqrt{\frac{1}{7} + \frac{1}{10}} + 2\sqrt{\frac{1}{7} + \frac{1}{18}} \\
&+ (4n - 10)\sqrt{\frac{1}{10} + \frac{1}{10}} + (8n - 20)\sqrt{\frac{1}{10} + \frac{1}{18}} \\
&+ (3n^2 - 16n + 21)\sqrt{\frac{1}{18} + \frac{1}{18}} \\
&= \frac{\sqrt{2}}{2}(6n^2 - 12n + 10) + \frac{4\sqrt{77}}{7} + \frac{\sqrt{35}}{10}(24n - 44) \\
&+ \frac{\sqrt{11}}{6}(12n^2 - 48n + 48) + \frac{2\sqrt{1190}}{35} + \frac{5\sqrt{14}}{21} + \frac{\sqrt{5}}{5}(4n - 10) \\
&+ \frac{\sqrt{35}}{15}(8n - 20) + \frac{1}{3}(3n^2 - 16n + 21) \\
&= (3\sqrt{2} + 2\sqrt{11} + 1)n^2 + (-6\sqrt{2} + \frac{44\sqrt{35}}{15} - 8\sqrt{11} + \frac{4\sqrt{5}}{5} - \frac{16}{3})n \\
&+ 5\sqrt{2} + \frac{4\sqrt{77}}{7} - \frac{86\sqrt{35}}{15} + 8\sqrt{11} + \frac{2\sqrt{1190}}{35} + \frac{5\sqrt{14}}{21} - 2\sqrt{5} + 7.
\end{aligned}
$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(RHDN3[n]) \\
&= \sum_{v_1v_2 \in E(RHDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}} \\
&= (6n^2 - 12n + 10)\frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4}}} + 8\frac{1}{\sqrt{\frac{1}{4} + \frac{1}{7}}} + (24n - 44)\frac{1}{\sqrt{\frac{1}{4} + \frac{1}{10}}} \\
&+ (12n^2 - 48n + 48)\frac{1}{\sqrt{\frac{1}{4} + \frac{1}{18}}} + 4\frac{1}{\sqrt{\frac{1}{7} + \frac{1}{10}}} + 2\frac{1}{\sqrt{\frac{1}{7} + \frac{1}{18}}} \\
&+ (4n - 10)\frac{1}{\sqrt{\frac{1}{10} + \frac{1}{10}}} + (8n - 20)\frac{1}{\sqrt{\frac{1}{10} + \frac{1}{18}}} \\
&+ (3n^2 - 16n + 21)\frac{1}{\sqrt{\frac{1}{18} + \frac{1}{18}}} \\
&= \sqrt{2}(6n^2 - 12n + 10) + \frac{16\sqrt{77}}{11} + \frac{2\sqrt{35}}{7}(24n - 44) \\
&+ \frac{6\sqrt{11}}{11}(12n^2 - 48n + 48) + \frac{4\sqrt{1190}}{17} + \frac{6\sqrt{14}}{5} + \sqrt{5}(4n - 10) \\
&+ \frac{3\sqrt{35}}{7}(8n - 20) + 3(3n^2 - 16n + 21)
\end{aligned}
$$

$$= (6\sqrt{2} + \frac{72\sqrt{11}}{11} + 9)n^2 + (-12\sqrt{2} + \frac{48\sqrt{35}}{7} - \frac{288\sqrt{11}}{11} + 4\sqrt{5} + \frac{24\sqrt{35}}{7} - 48)n + 10\sqrt{2} + \frac{16\sqrt{77}}{11} - \frac{88\sqrt{35}}{7} + \frac{288\sqrt{11}}{11} + \frac{4\sqrt{1190}}{17} + \frac{6\sqrt{14}}{5} - 10\sqrt{5} - \frac{60\sqrt{35}}{7} + 63.$$

□

Theorem 4.6 *Let SRHDN3[n] represents the subdivided rectangular hex-derived network of third type of dimension n. Then the Nirmala indices for the SRHDN3[n] network, where $n \geq 3$, is given by*

(i) $N(SRHDN3[n]) = (24\sqrt{6} + 36\sqrt{5})n^2 + (-48\sqrt{6} + 80\sqrt{3} - 144\sqrt{5})n + 32\sqrt{6} + 42 - 160\sqrt{3} + 144\sqrt{5}$,

(ii) $IN_1(SRHDN3[n]) = (12\sqrt{3} + 6\sqrt{5})n^2 + (-24\sqrt{3} + 8\sqrt{15} - 24\sqrt{5})n + 16\sqrt{3} + 3\sqrt{14} - 16\sqrt{15} + 24\sqrt{5}$,

(iii) $IN_2(SRHDN3[n]) = (16\sqrt{3} + \frac{54\sqrt{5}}{5})n^2 + (-32\sqrt{3} + \frac{40\sqrt{15}}{3} - \frac{216\sqrt{5}}{5})n + \frac{64\sqrt{3}}{3} + \frac{14\sqrt{14}}{3} - \frac{80\sqrt{15}}{3} + \frac{216\sqrt{5}}{5}$.

Proof: Table 7 details the edge set division of the *SRHDN*3[n] network depending on the degree of end vertices of each edge, which is essential for calculating the Nirmala indices of the network by employing their degree-dependent formulas, as illustrated below.

Table 7: The edge set partition of the *SRHDN*3[n] network based on the degree of end vertices of each edge [30].

Distinct pair of edges	$(2,4)$	$(2,7)$	$(2,10)$	$(2,18)$
Number of such pairs	$8(3n^2-6n+4)$	14	$40(n-2)$	$18(n-2)^2$

(*i*) **Nirmala Index:**

$$N(SRHDN3[n]) = \sum_{v_1v_2 \in E(SRHDN3[n])} \sqrt{d(v_1) + d(v_2)}$$
$$= 8(3n^2 - 6n + 4)\sqrt{2+4} + 14\sqrt{2+7} + 40(n-2)\sqrt{2+10}$$

$$
\begin{aligned}
&+ 18(n-2)^2\sqrt{2+18} \\
&= 8\sqrt{6}(3n^2-6n+4) + 42 + 80\sqrt{3}(n-2) + 36\sqrt{5}(n-2)^2 \\
&= (24\sqrt{6}+36\sqrt{5})n^2 + (-48\sqrt{6}+80\sqrt{3}-144\sqrt{5})n + 32\sqrt{6} + 42 \\
&\quad - 160\sqrt{3} + 144\sqrt{5}.
\end{aligned}
$$

(*ii*) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(SRHDN3[n]) \\
&= \sum_{v_1v_2 \in E(SRHDN3[n])} \sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}} \\
&= 8(3n^2-6n+4)\sqrt{\frac{1}{2}+\frac{1}{4}} + 14\sqrt{\frac{1}{2}+\frac{1}{7}} + 40(n-2)\sqrt{\frac{1}{2}+\frac{1}{10}} \\
&\quad + 18(n-2)^2\sqrt{\frac{1}{2}+\frac{1}{18}} \\
&= 4\sqrt{3}(3n^2-6n+4) + 3\sqrt{14} + 8\sqrt{15}(n-2) + 6\sqrt{5}(n-2)^2 \\
&= (12\sqrt{3}+6\sqrt{5})n^2 + (-24\sqrt{3}+8\sqrt{15}-24\sqrt{5})n + 16\sqrt{3} + 3\sqrt{14} \\
&\quad - 16\sqrt{15} + 24\sqrt{5}.
\end{aligned}
$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(SRHDN3[n]) \\
&= \sum_{v_1v_2 \in E(SRHDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}} \\
&= 8(3n^2-6n+4)\frac{1}{\sqrt{\frac{1}{2}+\frac{1}{4}}} + 14\frac{1}{\sqrt{\frac{1}{2}+\frac{1}{7}}} + 40(n-2)\frac{1}{\sqrt{\frac{1}{2}+\frac{1}{10}}} \\
&\quad + 18(n-2)^2\frac{1}{\sqrt{\frac{1}{2}+\frac{1}{18}}} \\
&= \frac{16\sqrt{3}}{3}(3n^2-6n+4) + \frac{14\sqrt{14}}{3} + \frac{40\sqrt{15}}{3}(n-2) + \frac{54\sqrt{5}}{5}(n-2)^2 \\
&= (16\sqrt{3}+\frac{54\sqrt{5}}{5})n^2 + (-32\sqrt{3}+\frac{40\sqrt{15}}{3}-\frac{216\sqrt{5}}{5})n + \frac{64\sqrt{3}}{3} \\
&\quad + \frac{14\sqrt{14}}{3} - \frac{80\sqrt{15}}{3} + \frac{216\sqrt{5}}{5}.
\end{aligned}
$$

□

Theorem 4.7 *Let CHDN3[n] represents the chain hex-derived network of third type of dimension n. Then the Nirmala indices for the CHDN3[n] network, where n ≥ 2, is given by*

(i) $N(CHDN3[n]) = (10\sqrt{2} + 12\sqrt{3} + 4)n + 12\sqrt{2} - 8\sqrt{3} - 8,$

(ii) $IN_1(CHDN3[n]) = (\frac{5\sqrt{2}}{2} + \frac{3\sqrt{6}}{2} + \frac{1}{2})n + 3\sqrt{2} - \sqrt{6} - 1,$

(iii) $IN_2(CHDN3[n]) = (5\sqrt{2} + 4\sqrt{6} + 2)n + 6\sqrt{2} - \frac{8\sqrt{6}}{3} - 4.$

Proof: Below, we evaluate the Nirmala indices of the *CHDN*3[n] network by employing their degree-dependent formulas, and the edge set partition (see Table 8) of the network.

Table 8: The edge set partition of the *CHDN*3[n] network based on the degree of end vertices of each edge [6].

Distinct pair of edges	$(4,4)$	$(4,8)$	$(8,8)$
Number of such pairs	$5n+6$	$6n-4$	$n-2$

(i) **Nirmala Index:**

$$\begin{aligned}
&N(CHDN3[n])\\
&= \sum_{v_1v_2 \in E(CHDN3[n])} \sqrt{d(v_1) + d(v_2)}\\
&= (5n+6)\sqrt{4+4} + (6n-4)\sqrt{4+8} + (n-2)\sqrt{8+8}\\
&= 2\sqrt{2}(5n+6) + 2\sqrt{3}(6n-4) + 4(n-2)\\
&= (10\sqrt{2} + 12\sqrt{3} + 4)n + 12\sqrt{2} - 8\sqrt{3} - 8.
\end{aligned}$$

(ii) **First Inverse Nirmala Index:**

$$\begin{aligned}
&IN_1(CHDN3[n])\\
&= \sum_{v_1v_2 \in E(CHDN3[n])} \sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}\\
&= (5n+6)\sqrt{\frac{1}{4}+\frac{1}{4}} + (6n-4)\sqrt{\frac{1}{4}+\frac{1}{8}} + (n-2)\sqrt{\frac{1}{8}+\frac{1}{8}}
\end{aligned}$$

$$= \frac{\sqrt{2}}{2}(5n+6) + \frac{\sqrt{6}}{4}(6n-4) + \frac{1}{2}(n-2)$$
$$= (\frac{5\sqrt{2}}{2} + \frac{3\sqrt{6}}{2} + \frac{1}{2})n + 3\sqrt{2} - \sqrt{6} - 1.$$

(*iii*) **Second Inverse Nirmala Index:**

$$IN_2(CHDN3[n])$$
$$= \sum_{v_1v_2 \in E(CHDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}}$$
$$= (5n+6)\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}} + (6n-4)\frac{1}{\sqrt{\frac{1}{4}+\frac{1}{8}}} + (n-2)\frac{1}{\sqrt{\frac{1}{8}+\frac{1}{8}}}$$
$$= \sqrt{2}(5n+6) + \frac{2\sqrt{6}}{3}(6n-4) + 2(n-2)$$
$$= (5\sqrt{2} + 4\sqrt{6} + 2)n + 6\sqrt{2} - \frac{8\sqrt{6}}{3} - 4.$$

□

Theorem 4.8 *Let SCHDN*3[n] *represents the subdivided chain hex-derived network of third type of dimension* n*. Then the Nirmala indices for the SCHDN*3[n] *network, where* $n \geq 2$*, is given by*

(i) $N(SCHDN3[n]) = (16\sqrt{6} + 8\sqrt{10})n + 8\sqrt{6} - 8\sqrt{10}$,

(ii) $IN_1(SCHDN3[n]) = (8\sqrt{3} + 2\sqrt{10})n + 4\sqrt{3} - 2\sqrt{10}$,

(iii) $IN_2(SCHDN3[n]) = (\frac{32\sqrt{3}}{3} + \frac{16\sqrt{10}}{5})n + \frac{16\sqrt{3}}{3} - \frac{16\sqrt{10}}{5}$.

Proof: Depending on the degree of end vertices of each edge, Table 9 displays the edge set partition of the *SCHDN*3[n] network that is necessary to compute the Nirmala indices of the network by using their degree-dependent formulas, as depicted below.

Table 9: The edge set partition of the *SCHDN*3[n] network based on the degree of end vertices of each edge [29].

Distinct pair of edges	$(2,4)$	$(2,8)$
Number of such pairs	$16n+8$	$8n-8$

(i) **Nirmala Index:**

$$\begin{aligned}
&N(SCHDN3[n])\\
&= \sum_{v_1v_2 \in E(SCHDN3[n])} \sqrt{d(v_1) + d(v_2)}\\
&= (16n+8)\sqrt{2+4} + (8n-8)\sqrt{2+8}\\
&= \sqrt{6}(16n+8) + \sqrt{10}(8n-8)\\
&= (16\sqrt{6} + 8\sqrt{10})n + 8\sqrt{6} - 8\sqrt{10}.
\end{aligned}$$

(ii) **First Inverse Nirmala Index:**

$$\begin{aligned}
&IN_1(SCHDN3[n])\\
&= \sum_{v_1v_2 \in E(SCHDN3[n])} \sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}\\
&= (16n+8)\sqrt{\frac{1}{2} + \frac{1}{4}} + (8n-8)\sqrt{\frac{1}{2} + \frac{1}{8}}\\
&= \frac{\sqrt{3}}{2}(16n+8) + \frac{\sqrt{10}}{4}(8n-8)\\
&= (8\sqrt{3} + 2\sqrt{10})n + 4\sqrt{3} - 2\sqrt{10}.
\end{aligned}$$

(iii) **Second Inverse Nirmala Index:**

$$\begin{aligned}
&IN_2(SCHDN3[n])\\
&= \sum_{v_1v_2 \in E(SCHDN3[n])} \frac{1}{\sqrt{\frac{1}{d(v_1)} + \frac{1}{d(v_2)}}}\\
&= (16n+8)\frac{1}{\sqrt{\frac{1}{2} + \frac{1}{4}}} + (8n-8)\frac{1}{\sqrt{\frac{1}{2} + \frac{1}{8}}}\\
&= \frac{2\sqrt{3}}{3}(16n+8) + \frac{2\sqrt{10}}{5}(8n-8)\\
&= (\frac{32\sqrt{3}}{3} + \frac{16\sqrt{10}}{5})n + \frac{16\sqrt{3}}{3} - \frac{16\sqrt{10}}{5}.
\end{aligned}$$

□

4.2 Determining the Nirmala Indices Using the M-polynomial Approach

In this section, we start by summarizing the M-polynomial expressions of the considered hex-derived networks of type three and their subdivisions, which have been derived in [5–7, 9, 10, 29, 30]. Using these M-polynomial expressions of the networks taken into account, we calculate the Nirmala index, the first inverse Nirmala index and the second inverse Nirmala index for them through their M-polynomial based formulas listed in Table 1.

Theorem 4.9 ([5, 10]) *Let $HDN3[n]$ and $SHDN3[n]$ denote the hex-derived and subdivided hex-derived network of third type, then the expressions of their M-polynomials are given by*

(i) $M(HDN3[n];x,y) = (18n^2-36n+18)x^4y^4+24x^4y^7+(36n-72)x^4y^{10}+(36n^2-108n+84)x^4y^{18}+12x^7y^{10}+6x^7y^{18}+(6n-18)x^{10}y^{10}+(12n-24)x^{10}y^{18}+(9n^2-33n+30)x^{18}y^{18}$,

(ii) $M(SHDN3[n];x,y) = 72(n-1)^2x^2y^4+42x^2y^7+60(n-2)x^2y^{10}+18(3n^2-9n+7)x^2y^{18}$.

Theorem 4.10 ([7, 30]) *Let $THDN3[n]$ and $STHDN3[n]$ stand for the triangular hex-derived and subdivided triangular hex-derived network of third type, then the expressions of their M-polynomials are given by*

(i) $M(THDN3[n];x,y) = (3n^2-6n+9)x^4y^4+(18n-30)x^4y^{10}+(6n^2-30n+36)x^4y^{18}+(3n-6)x^{10}y^{10}+(6n-18)x^{10}y^{18}+\frac{1}{2}(3n^2-21n+36)x^{18}y^{18}$,

(ii) $M(STHDN3[n];x,y) = 12(n^2-2n+2)x^2y^4+30(n-2)x^2y^{10}+9(n-2)(n-3)x^2y^{18}$.

Theorem 4.11 ([9, 30]) *Let $RHDN3[n]$ and $SRHDN3[n]$ represent the rectangular hex-derived and subdivided rectangular hex-derived network of third type, then the expressions of their M-polynomials are given by*

(i) $M(RHDN3[n];x,y) = (6n^2-12n+10)x^4y^4+8x^4y^7+(24n-44)x^4y^{10}+(12n^2-48n+48)x^4y^{18}+4x^7y^{10}+2x^7y^{18}+(4n-10)x^{10}y^{10}+(8n-20)x^{10}y^{18}+(3n^2-16n+21)x^{18}y^{18}$,

(ii) $M(SRHDN3[n];x,y) = 8(3n^2-6n+4)x^2y^4+14x^2y^7+40(n-2)x^2y^{10}+18(n-2)^2x^2y^{18}$.

Theorem 4.12 ([6, 29]) *Let CHDN3[n] and SCHDN3[n] be the chain hex-derived and subdivided chain hex-derived network of third type, then the expressions of their M-polynomials are given by*

(i) $M(CHDN3[n]; x, y) = (5n+6)x^4y^4 + (6n-4)x^4y^8 + (n-2)x^8y^8$,

(ii) $M(SCHDN3[n]; x, y) = (16n+8)x^2y^4 + (8n-8)x^2y^8$.

Let us now proof the Theorems 4.1 to 4.8 to calculate the Nirmala indices of our considered networks using their M-polynomials listed in Theorems 4.9 to 4.12.

Proof: (**Another (M-polynomial based) approach to proving Theorems 4.1 to 4.8**). Now, with the help of the above-mentioned M-polynomial expressions of the networks, below we can calculate the Nirmala indices for the considered networks using the formulas listed in Table 1 (on Page 195).

Proof of Theorem 4.1 (The Nirmala Indices for the *HDN*3[*n*] Network).

(*i*) **Nirmala Index:**

$$\begin{aligned}
&N(HDN3[n])\\
&= D_x^{1/2} JM(HDN3[n]; x, y)|_{x=1}\\
&= D_x^{1/2} J\{(18n^2-36n+18)x^4y^4 + 24x^4y^7 + (36n-72)x^4y^{10}\\
&\quad + (36n^2-108n+84)x^4y^{18} + 12x^7y^{10} + 6x^7y^{18} + (6n-18)x^{10}y^{10}\\
&\quad + (12n-24)x^{10}y^{18} + (9n^2-33n+30)x^{18}y^{18}\}|_{x=1}\\
&= D_x^{1/2}\{(18n^2-36n+18)x^8 + 24x^{11} + (36n-72)x^{14}\\
&\quad + (36n^2-108n+84)x^{22} + 12x^{17} + 6x^{25} + (6n-18)x^{20}\\
&\quad + (12n-24)x^{28} + (9n^2-33n+30)x^{36}\}|_{x=1}\\
&= \sqrt{8}\times(18n^2-36n+18) + \sqrt{11}\times 24 + \sqrt{14}\times(36n-72)\\
&\quad + \sqrt{22}\times(36n^2-108n+84) + \sqrt{17}\times 12 + \sqrt{25}\times 6\\
&\quad + \sqrt{20}\times(6n-18) + \sqrt{28}\times(12n-24) + \sqrt{36}\times(9n^2-33n+30)\\
&= (36\sqrt{2}+36\sqrt{22}+54)n^2 + (-72\sqrt{2}+36\sqrt{14}-108\sqrt{22}+12\sqrt{5}\\
&\quad + 24\sqrt{7}-198)n + 36\sqrt{2}+24\sqrt{11}-72\sqrt{14}+84\sqrt{22}+12\sqrt{17}\\
&\quad + 210 - 36\sqrt{5} - 48\sqrt{7}.
\end{aligned}$$

(*ii*) **First Inverse Nirmala Index:**

$IN_1(HDN3[n])$

$$
\begin{aligned}
&= D_x^{1/2} J S_y^{1/2} S_x^{1/2} M(HDN3[n]; x, y)|_{x=1} \\
&= D_x^{1/2} J S_y^{1/2} S_x^{1/2} \{(18n^2 - 36n + 18)x^4y^4 + 24x^4y^7 + (36n - 72)x^4y^{10} \\
&\quad + (36n^2 - 108n + 84)x^4y^{18} + 12x^7y^{10} + 6x^7y^{18} + (6n - 18)x^{10}y^{10} \\
&\quad + (12n - 24)x^{10}y^{18} + (9n^2 - 33n + 30)x^{18}y^{18}\}|_{x=1} \\
&= D_x^{1/2} J \Big\{ \frac{1}{\sqrt{4 \times 4}}(18n^2 - 36n + 18)x^4y^4 + \frac{1}{\sqrt{4 \times 7}}24x^4y^7 \\
&\quad + \frac{1}{\sqrt{4 \times 10}}(36n - 72)x^4y^{10} + \frac{1}{\sqrt{4 \times 18}}(36n^2 - 108n + 84)x^4y^{18} \\
&\quad + \frac{1}{\sqrt{7 \times 10}}12x^7y^{10} + \frac{1}{\sqrt{7 \times 18}}6x^7y^{18} + \frac{1}{\sqrt{10 \times 10}}(6n - 18)x^{10}y^{10} \\
&\quad + \frac{1}{\sqrt{10 \times 18}}(12n - 24)x^{10}y^{18} \\
&\quad + \frac{1}{\sqrt{18 \times 18}}(9n^2 - 33n + 30)x^{18}y^{18} \Big\}\Big|_{x=1} \\
&= D_x^{1/2} \Big\{ \frac{1}{4}(18n^2 - 36n + 18)x^8 + \frac{1}{\sqrt{28}}24x^{11} + \frac{1}{\sqrt{40}}(36n - 72)x^{14} \\
&\quad + \frac{1}{\sqrt{72}}(36n^2 - 108n + 84)x^{22} + \frac{1}{\sqrt{70}}12x^{17} + \frac{1}{\sqrt{126}}6x^{25} \\
&\quad + \frac{1}{10}(6n - 18)x^{20} + \frac{1}{\sqrt{180}}(12n - 24)x^{28} \\
&\quad + \frac{1}{18}(9n^2 - 33n + 30)x^{36} \Big\}\Big|_{x=1} \\
&= \frac{\sqrt{8}}{4}(18n^2 - 36n + 18) + \frac{\sqrt{11}}{\sqrt{28}} \times 24 + \frac{\sqrt{14}}{\sqrt{40}}(36n - 72) \\
&\quad + \frac{\sqrt{22}}{\sqrt{72}}(36n^2 - 108n + 84) + \frac{\sqrt{17}}{\sqrt{70}} \times 12 + \frac{\sqrt{25}}{\sqrt{126}} \times 6 \\
&\quad + \frac{\sqrt{20}}{10}(6n - 18) + \frac{\sqrt{28}}{\sqrt{180}}(12n - 24) + \frac{\sqrt{36}}{18}(9n^2 - 33n + 30) \\
&= (9\sqrt{2} + 6\sqrt{11} + 3)n^2 + (-18\sqrt{2} + \frac{22\sqrt{35}}{5} - 18\sqrt{11} + \frac{6\sqrt{5}}{5} - 11)n \\
&\quad + 9\sqrt{2} + \frac{12\sqrt{77}}{7} - \frac{44\sqrt{35}}{5} + 14\sqrt{11} + \frac{6\sqrt{1190}}{35} + \frac{5\sqrt{14}}{7} \\
&\quad - \frac{18\sqrt{5}}{5} + 10.
\end{aligned}
$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(HDN3[n])\\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} M(HDN3[n]; x, y)|_{x=1}\\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} \{(18n^2-36n+18)x^4y^4 + 24x^4y^7\\
&\quad + (36n-72)x^4y^{10} + (36n^2-108n+84)x^4y^{18} + 12x^7y^{10} + 6x^7y^{18}\\
&\quad + (6n-18)x^{10}y^{10} + (12n-24)x^{10}y^{18}\\
&\quad + (9n^2-33n+30)x^{18}y^{18}\}|_{x=1}\\
&= S_x^{1/2} J \{\sqrt{4\times 4}\times(18n^2-36n+18)x^4y^4 + \sqrt{4\times 7}\times 24x^4y^7\\
&\quad + \sqrt{4\times 10}\times(36n-72)x^4y^{10} + \sqrt{4\times 18}\times(36n^2-108n+84)x^4y^{18}\\
&\quad + \sqrt{7\times 10}\times 12x^7y^{10} + \sqrt{7\times 18}\times 6x^7y^{18}\\
&\quad + \sqrt{10\times 10}\times(6n-18)x^{10}y^{10} + \sqrt{10\times 18}\times(12n-24)x^{10}y^{18}\\
&\quad + \sqrt{18\times 18}\times(9n^2-33n+30)x^{18}y^{18}\}|_{x=1}\\
&= S_x^{1/2} \{4\times(18n^2-36n+18)x^8 + \sqrt{28}\times 24x^{11}\\
&\quad + \sqrt{40}\times(36n-72)x^{14} + \sqrt{72}\times(36n^2-108n+84)x^{22}\\
&\quad + \sqrt{70}\times 12x^{17} + \sqrt{126}\times 6x^{25} + 10\times(6n-18)x^{20}\\
&\quad + \sqrt{180}\times(12n-24)x^{28} + 18\times(9n^2-33n+30)x^{36}\}|_{x=1}\\
&= \frac{4}{\sqrt{8}}\times(18n^2-36n+18) + \frac{\sqrt{28}}{\sqrt{11}}\times 24 + \frac{\sqrt{40}}{\sqrt{14}}\times(36n-72)\\
&\quad + \frac{\sqrt{72}}{\sqrt{22}}\times(36n^2-108n+84) + \frac{\sqrt{70}}{\sqrt{17}}\times 12 + \frac{\sqrt{126}}{\sqrt{25}}\times 6\\
&\quad + \frac{10}{\sqrt{20}}\times(6n-18) + \frac{\sqrt{180}}{\sqrt{28}}\times(12n-24)\\
&\quad + \frac{18}{\sqrt{36}}\times(9n^2-33n+30)\\
&= (18\sqrt{2} + \frac{216\sqrt{11}}{11} + 27)n^2 + (-36\sqrt{2} + \frac{72\sqrt{35}}{7} - \frac{648\sqrt{11}}{11}\\
&\quad + 6\sqrt{5} + \frac{36\sqrt{35}}{7} - 99)n + 18\sqrt{2} + \frac{48\sqrt{77}}{11} - \frac{144\sqrt{35}}{7}\\
&\quad + \frac{504\sqrt{11}}{11} + \frac{12\sqrt{1190}}{17} + \frac{18\sqrt{14}}{5} - 18\sqrt{5} - \frac{72\sqrt{35}}{7} + 90.
\end{aligned}
$$

Proof of Theorem 4.2 (The Nirmala Indices for the *SHDN*3[*n*]

Network).

(*i*) **Nirmala Index:**

$$\begin{aligned}
&N(SHDN3[n])\\
&= D_x^{1/2} JM(SHDN3[n]; x, y)|_{x=1}\\
&= D_x^{1/2} J\{72(n-1)^2x^2y^4 + 42x^2y^7 + 60(n-2)x^2y^{10}\\
&\quad + 18(3n^2-9n+7)x^2y^{18}\}|_{x=1}\\
&= D_x^{1/2}\{72(n-1)^2x^6 + 42x^9 + 60(n-2)x^{12}\\
&\quad + 18(3n^2-9n+7)x^{20}\}|_{x=1}\\
&= \sqrt{6}\times 72(n-1)^2 + \sqrt{9}\times 42 + \sqrt{12}\times 60(n-2)\\
&\quad + \sqrt{20}\times 18(3n^2-9n+7)\\
&= (72\sqrt{6}+108\sqrt{5})n^2 + (-144\sqrt{6}+120\sqrt{3}-324\sqrt{5})n + 72\sqrt{6}\\
&\quad + 126 - 240\sqrt{3} + 252\sqrt{5}.
\end{aligned}$$

(*ii*) **First Inverse Nirmala Index:**

$$\begin{aligned}
&IN_1(SHDN3[n])\\
&= D_x^{1/2} JS_y^{1/2} S_x^{1/2} M(SHDN3[n]; x, y)|_{x=1}\\
&= D_x^{1/2} JS_y^{1/2} S_x^{1/2}\{72(n-1)^2x^2y^4 + 42x^2y^7 + 60(n-2)x^2y^{10}\\
&\quad + 18(3n^2-9n+7)x^2y^{18}\}|_{x=1}\\
&= D_x^{1/2} J\Big\{\frac{1}{\sqrt{2\times 4}}72(n-1)^2x^2y^4 + \frac{1}{\sqrt{2\times 7}}42x^2y^7\\
&\quad + \frac{1}{\sqrt{2\times 10}}60(n-2)x^2y^{10}\\
&\quad + \frac{1}{\sqrt{2\times 18}}18(3n^2-9n+7)x^2y^{18}\Big\}\Big|_{x=1}\\
&= D_x^{1/2}\Big\{\frac{1}{\sqrt{8}}72(n-1)^2x^6 + \frac{1}{\sqrt{14}}42x^9 + \frac{1}{\sqrt{20}}60(n-2)x^{12}\\
&\quad + \frac{1}{\sqrt{36}}18(3n^2-9n+7)x^{20}\Big\}\Big|_{x=1}\\
&= \frac{\sqrt{6}}{\sqrt{8}}72(n-1)^2 + \frac{\sqrt{9}}{\sqrt{14}}42 + \frac{\sqrt{12}}{\sqrt{20}}60(n-2) + \frac{\sqrt{20}}{\sqrt{36}}18(3n^2-9n+7)\\
&= (36\sqrt{3}+18\sqrt{5})n^2 + (-72\sqrt{3}+12\sqrt{15}-54\sqrt{5})n + 36\sqrt{3} + 9\sqrt{14}\\
&\quad - 24\sqrt{15} + 42\sqrt{5}.
\end{aligned}$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(SHDN3[n])\\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} M(SHDN3[n]; x, y)|_{x=1}\\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} \{72(n-1)^2 x^2 y^4 + 42x^2 y^7 + 60(n-2)x^2 y^{10}\\
&\quad + 18(3n^2 - 9n + 7)x^2 y^{18}\}|_{x=1}\\
&= S_x^{1/2} J \{\sqrt{2\times 4} \times 72(n-1)^2 x^2 y^4 + \sqrt{2\times 7} \times 42x^2 y^7\\
&\quad + \sqrt{2\times 10} \times 60(n-2)x^2 y^{10}\\
&\quad + \sqrt{2\times 18} \times 18(3n^2 - 9n + 7)x^2 y^{18}\}\big|_{x=1}\\
&= S_x^{1/2} \{\sqrt{8} \times 72(n-1)^2 x^6 + \sqrt{14} \times 42x^9 + \sqrt{20} \times 60(n-2)x^{12}\\
&\quad + \sqrt{36} \times 18(3n^2 - 9n + 7)x^{20}\}\big|_{x=1}\\
&= \frac{\sqrt{8}}{\sqrt{6}} \times 72(n-1)^2 + \frac{\sqrt{14}}{\sqrt{9}} \times 42 + \frac{\sqrt{20}}{\sqrt{12}} \times 60(n-2)\\
&\quad + \frac{\sqrt{36}}{\sqrt{20}} \times 18(3n^2 - 9n + 7)\\
&= (48\sqrt{3} + \frac{162\sqrt{5}}{5})n^2 + (-96\sqrt{3} + 20\sqrt{15} - \frac{486\sqrt{5}}{5})n + 48\sqrt{3}\\
&\quad + 14\sqrt{14} - 40\sqrt{15} + \frac{378\sqrt{5}}{5}.
\end{aligned}
$$

Proof of Theorem 4.3 (The Nirmala Indices for the *THDN*3[*n*] Network).

(*i*) **Nirmala Index:**

$$
\begin{aligned}
&N(THDN3[n])\\
&= D_x^{1/2} J M(THDN3[n]; x, y)|_{x=1}\\
&= D_x^{1/2} J \{(3n^2 - 6n + 9)x^4 y^4 + (18n - 30)x^4 y^{10}\\
&\quad + (6n^2 - 30n + 36)x^4 y^{18} + (3n - 6)x^{10} y^{10} + (6n - 18)x^{10} y^{18}\\
&\quad + \frac{1}{2}(3n^2 - 21n + 36)x^{18} y^{18}\}|_{x=1}\\
&= D_x^{1/2} \{(3n^2 - 6n + 9)x^8 + (18n - 30)x^{14} + (6n^2 - 30n + 36)x^{22}\\
&\quad + (3n - 6)x^{20} + (6n - 18)x^{28} + \frac{1}{2}(3n^2 - 21n + 36)x^{36}\}|_{x=1}\\
&= \sqrt{8} \times (3n^2 - 6n + 9) + \sqrt{14} \times (18n - 30)
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{22}\times(6n^2-30n+36)+\sqrt{20}\times(3n-6)+\sqrt{28}\times(6n-18)\\
&+\frac{\sqrt{36}}{2}\times(3n^2-21n+36)\\
&=(6\sqrt{2}+6\sqrt{22}+9)n^2+(-12\sqrt{2}+18\sqrt{14}-30\sqrt{22}+6\sqrt{5}\\
&+12\sqrt{7}-63)n+18\sqrt{2}-30\sqrt{14}+36\sqrt{22}-12\sqrt{5}-36\sqrt{7}+108.
\end{aligned}
$$

(*ii*) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(THDN3[n])\\
&=D_x^{1/2}JS_y^{1/2}S_x^{1/2}M(THDN3[n];x,y)|_{x=1}\\
&=D_x^{1/2}JS_y^{1/2}S_x^{1/2}\{(3n^2-6n+9)x^4y^4+(18n-30)x^4y^{10}\\
&+(6n^2-30n+36)x^4y^{18}+(3n-6)x^{10}y^{10}+(6n-18)x^{10}y^{18}\\
&+\frac{1}{2}(3n^2-21n+36)x^{18}y^{18}\}|_{x=1}\\
&=D_x^{1/2}J\Big\{\frac{1}{\sqrt{4\times4}}(3n^2-6n+9)x^4y^4+\frac{1}{\sqrt{4\times10}}(18n-30)x^4y^{10}\\
&+\frac{1}{\sqrt{4\times18}}(6n^2-30n+36)x^4y^{18}+\frac{1}{\sqrt{10\times10}}(3n-6)x^{10}y^{10}\\
&+\frac{1}{\sqrt{10\times18}}(6n-18)x^{10}y^{18}\\
&+\frac{1}{2\sqrt{18\times18}}(3n^2-21n+36)x^{18}y^{18}\Big\}\Big|_{x=1}\\
&=D_x^{1/2}\Big\{\frac{1}{4}(3n^2-6n+9)x^8+\frac{1}{\sqrt{40}}(18n-30)x^{14}\\
&+\frac{1}{\sqrt{72}}(6n^2-30n+36)x^{22}+\frac{1}{10}(3n-6)x^{20}+\frac{1}{\sqrt{180}}(6n-18)x^{28}\\
&+\frac{1}{36}(3n^2-21n+36)x^{36}\Big\}\Big|_{x=1}\\
&=\frac{\sqrt{8}}{4}(3n^2-6n+9)+\frac{\sqrt{14}}{\sqrt{40}}(18n-30)+\frac{\sqrt{22}}{\sqrt{72}}(6n^2-30n+36)\\
&+\frac{\sqrt{20}}{10}(3n-6)+\frac{\sqrt{28}}{\sqrt{180}}(6n-18)+\frac{\sqrt{36}}{36}(3n^2-21n+36)\\
&=(\frac{3\sqrt{2}}{2}+\sqrt{11}+\frac{1}{2})n^2+(-3\sqrt{2}+\frac{11\sqrt{35}}{5}-5\sqrt{11}+\frac{3\sqrt{5}}{5}-\frac{7}{2})n\\
&+\frac{9\sqrt{2}}{2}-\frac{21\sqrt{35}}{5}+6\sqrt{11}-\frac{6\sqrt{5}}{5}+6.
\end{aligned}
$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(THDN3[n])\\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} M(THDN3[n]; x, y)|_{x=1}\\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} \{(3n^2-6n+9)x^4y^4 + (18n-30)x^4y^{10}\\
&\quad + (6n^2-30n+36)x^4y^{18} + (3n-6)x^{10}y^{10} + (6n-18)x^{10}y^{18}\\
&\quad + \frac{1}{2}(3n^2-21n+36)x^{18}y^{18}\}|_{x=1}\\
&= S_x^{1/2} J\{\sqrt{4\times 4}\times(3n^2-6n+9)x^4y^4 + \sqrt{4\times 10}\times(18n-30)x^4y^{10}\\
&\quad + \sqrt{4\times 18}\times(6n^2-30n+36)x^4y^{18} + \sqrt{10\times 10}\times(3n-6)x^{10}y^{10}\\
&\quad + \sqrt{10\times 18}\times(6n-18)x^{10}y^{18}\\
&\quad + \sqrt{18\times 18}\times\frac{1}{2}(3n^2-21n+36)x^{18}y^{18}\}|_{x=1}\\
&= S_x^{1/2}\{4\times(3n^2-6n+9)x^8 + \sqrt{40}\times(18n-30)x^{14}\\
&\quad + \sqrt{72}\times(6n^2-30n+36)x^{22} + 10\times(3n-6)x^{20}\\
&\quad + \sqrt{180}\times(6n-18)x^{28} + 9(3n^2-21n+36)x^{36}\}|_{x=1}\\
&= \frac{4}{\sqrt{8}}\times(3n^2-6n+9) + \frac{\sqrt{40}}{\sqrt{14}}\times(18n-30)\\
&\quad + \frac{\sqrt{72}}{\sqrt{22}}\times(6n^2-30n+36) + \frac{10}{\sqrt{20}}\times(3n-6)\\
&\quad + \frac{\sqrt{180}}{\sqrt{28}}\times(6n-18) + \frac{9}{\sqrt{36}}\times(3n^2-21n+36)\\
&= (3\sqrt{2} + \frac{36\sqrt{11}}{11} + \frac{9}{2})n^2 + (-6\sqrt{2} + \frac{36\sqrt{35}}{7} - \frac{180\sqrt{11}}{11} + 3\sqrt{5}\\
&\quad + \frac{18\sqrt{35}}{7} - \frac{63}{2})n + 9\sqrt{2} - \frac{60\sqrt{35}}{7} + \frac{216\sqrt{11}}{11} - 6\sqrt{5} - \frac{54\sqrt{35}}{7} + 54.
\end{aligned}
$$

Proof of Theorem 4.4 (The Nirmala Indices for the *STHDN*3[*n*] Network).

(*i*) **Nirmala Index:**

$$
\begin{aligned}
&N(STHDN3[n])\\
&= D_x^{1/2} J M(STHDN3[n]; x, y)|_{x=1}\\
&= D_x^{1/2} J\{12(n^2-2n+2)x^2y^4 + 30(n-2)x^2y^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+ 9(n-2)(n-3)x^2y^{18}\}|_{x=1}\\
&= D_x^{1/2}\{12(n^2-2n+2)x^6 + 30(n-2)x^{12} + 9(n-2)(n-3)x^{20}\}|_{x=1}\\
&= \sqrt{6}\times 12(n^2-2n+2) + \sqrt{12}\times 30(n-2) + \sqrt{20}\times 9(n-2)(n-3)\\
&= (12\sqrt{6}+18\sqrt{5})n^2 + (-24\sqrt{6}+60\sqrt{3}-90\sqrt{5})n + 24\sqrt{6}\\
&\quad - 120\sqrt{3} + 108\sqrt{5}.
\end{aligned}
$$

(*ii*) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(STHDN3[n])\\
&= D_x^{1/2}JS_y^{1/2}S_x^{1/2}M(STHDN3[n];x,y)|_{x=1}\\
&= D_x^{1/2}JS_y^{1/2}S_x^{1/2}\{12(n^2-2n+2)x^2y^4 + 30(n-2)x^2y^{10}\\
&\quad + 9(n-2)(n-3)x^2y^{18}\}|_{x=1}\\
&= D_x^{1/2}J\Big\{\frac{1}{\sqrt{2\times 4}}12(n^2-2n+2)x^2y^4 + \frac{1}{\sqrt{2\times 10}}30(n-2)x^2y^{10}\\
&\quad + \frac{1}{\sqrt{2\times 18}}9(n-2)(n-3)x^2y^{18}\Big\}\Big|_{x=1}\\
&= D_x^{1/2}\Big\{\frac{1}{\sqrt{8}}12(n^2-2n+2)x^6 + \frac{1}{\sqrt{20}}30(n-2)x^{12}\\
&\quad + \frac{1}{\sqrt{36}}9(n-2)(n-3)x^{20}\Big\}\Big|_{x=1}\\
&= \frac{\sqrt{6}}{\sqrt{8}}\times 12(n^2-2n+2) + \frac{\sqrt{12}}{\sqrt{20}}\times 30(n-2)\\
&\quad + \frac{\sqrt{20}}{\sqrt{36}}\times 9(n-2)(n-3)\\
&= (6\sqrt{3}+3\sqrt{5})n^2 + (-12\sqrt{3}+6\sqrt{15}-15\sqrt{5})n\\
&\quad + 12\sqrt{3} - 12\sqrt{15} + 18\sqrt{5}.
\end{aligned}
$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(STHDN3[n])\\
&= S_x^{1/2}JD_y^{1/2}D_x^{1/2}M(STHDN3[n];x,y)|_{x=1}\\
&= S_x^{1/2}JD_y^{1/2}D_x^{1/2}\{12(n^2-2n+2)x^2y^4 + 30(n-2)x^2y^{10}\\
&\quad + 9(n-2)(n-3)x^2y^{18}\}|_{x=1}\\
&= S_x^{1/2}J\{\sqrt{2\times 4}\times 12(n^2-2n+2)x^2y^4 + \sqrt{2\times 10}\times 30(n-2)x^2y^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{2\times 18}\times 9(n-2)(n-3)x^2y^{18}\}\big|_{x=1}\\
&=S_x^{1/2}\{\sqrt{8}\times 12(n^2-2n+2)x^6+\sqrt{20}\times 30(n-2)x^{12}\\
&+\sqrt{36}\times 9(n-2)(n-3)x^{20}\}\big|_{x=1}\\
&=\frac{\sqrt{8}}{\sqrt{6}}\times 12(n^2-2n+2)+\frac{\sqrt{20}}{\sqrt{12}}\times 30(n-2)+\frac{\sqrt{36}}{\sqrt{20}}\times 9(n-2)(n-3)\\
&=(8\sqrt{3}+\frac{27\sqrt{5}}{5})n^2+(-16\sqrt{3}+10\sqrt{15}-27\sqrt{5})n+16\sqrt{3}-20\sqrt{15}\\
&+\frac{162\sqrt{5}}{5}.
\end{aligned}
$$

Proof of Theorem 4.5 (The Nirmala Indices for the *RHDN*3[*n*] Network).

(*i*) **Nirmala Index:**

$$
\begin{aligned}
&N(RHDN3[n])\\
&=D_x^{1/2}JM(RHDN3[n];x,y)|_{x=1}\\
&=D_x^{1/2}J\{(6n^2-12n+10)x^4y^4+8x^4y^7+(24n-44)x^4y^{10}\\
&+(12n^2-48n+48)x^4y^{18}+4x^7y^{10}+2x^7y^{18}+(4n-10)x^{10}y^{10}\\
&+(8n-20)x^{10}y^{18}+(3n^2-16n+21)x^{18}y^{18}\}|_{x=1}\\
&=D_x^{1/2}\{(6n^2-12n+10)x^8+8x^{11}+(24n-44)x^{14}\\
&+(12n^2-48n+48)x^{22}+4x^{17}+2x^{25}+(4n-10)x^{20}\\
&+(8n-20)x^{28}+(3n^2-16n+21)x^{36}\}|_{x=1}\\
&=\sqrt{8}\times(6n^2-12n+10)+\sqrt{11}\times 8+\sqrt{14}\times(24n-44)\\
&+\sqrt{22}\times(12n^2-48n+48)+\sqrt{17}\times 4+\sqrt{25}\times 2\\
&+\sqrt{20}\times(4n-10)+\sqrt{28}\times(8n-20)+\sqrt{36}\times(3n^2-16n+21)\\
&=(12\sqrt{2}+12\sqrt{22}+18)n^2+(-24\sqrt{2}+24\sqrt{14}-48\sqrt{22}+8\sqrt{5}\\
&+16\sqrt{7}-96)n+20\sqrt{2}+8\sqrt{11}-44\sqrt{14}+48\sqrt{22}+4\sqrt{17}\\
&+136-20\sqrt{5}-40\sqrt{7}.
\end{aligned}
$$

(*ii*) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(RHDN3[n])\\
&=D_x^{1/2}JS_y^{1/2}S_x^{1/2}M(RHDN3[n];x,y)|_{x=1}
\end{aligned}
$$

$$
\begin{aligned}
&= D_x^{1/2} J S_y^{1/2} S_x^{1/2} \{(6n^2-12n+10)x^4y^4 + 8x^4y^7 + (24n-44)x^4y^{10} \\
&\quad + (12n^2-48n+48)x^4y^{18} + 4x^7y^{10} + 2x^7y^{18} + (4n-10)x^{10}y^{10} \\
&\quad + (8n-20)x^{10}y^{18} + (3n^2-16n+21)x^{18}y^{18}\}|_{x=1} \\
&= D_x^{1/2} J\Big\{\frac{1}{\sqrt{4\times 4}}(6n^2-12n+10)x^4y^4 + \frac{1}{\sqrt{4\times 7}}8x^4y^7 \\
&\quad + \frac{1}{\sqrt{4\times 10}}(24n-44)x^4y^{10} + \frac{1}{\sqrt{4\times 18}}(12n^2-48n+48)x^4y^{18} \\
&\quad + \frac{1}{\sqrt{7\times 10}}4x^7y^{10} + \frac{1}{\sqrt{7\times 18}}2x^7y^{18} + \frac{1}{\sqrt{10\times 10}}(4n-10)x^{10}y^{10} \\
&\quad + \frac{1}{\sqrt{10\times 18}}(8n-20)x^{10}y^{18} \\
&\quad + \frac{1}{\sqrt{18\times 18}}(3n^2-16n+21)x^{18}y^{18}\Big\}\Big|_{x=1} \\
&= D_x^{1/2}\Big\{\frac{1}{4}(6n^2-12n+10)x^8 + \frac{1}{\sqrt{28}}\times 8x^{11} + \frac{1}{\sqrt{40}}(24n-44)x^{14} \\
&\quad + \frac{1}{\sqrt{72}}(12n^2-48n+48)x^{22} + \frac{1}{\sqrt{70}}\times 4x^{17} + \frac{1}{\sqrt{126}}\times 2x^{25} \\
&\quad + \frac{1}{10}(4n-10)x^{20} + \frac{1}{\sqrt{180}}(8n-20)x^{28} \\
&\quad + \frac{1}{18}(3n^2-16n+21)x^{36}\Big\}\Big|_{x=1} \\
&= \frac{\sqrt{8}}{4}(6n^2-12n+10) + \frac{\sqrt{11}}{\sqrt{28}}\times 8 + \frac{\sqrt{14}}{\sqrt{40}}(24n-44) \\
&\quad + \frac{\sqrt{22}}{\sqrt{72}}(12n^2-48n+48) + \frac{\sqrt{17}}{\sqrt{70}}\times 4 + \frac{\sqrt{25}}{\sqrt{126}}\times 2 \\
&\quad + \frac{\sqrt{20}}{10}(4n-10) + \frac{\sqrt{28}}{\sqrt{180}}(8n-20) + \frac{\sqrt{36}}{18}(3n^2-16n+21) \\
&= (3\sqrt{2}+2\sqrt{11}+1)n^2 + (-6\sqrt{2} + \frac{44\sqrt{35}}{15} - 8\sqrt{11} + \frac{4\sqrt{5}}{5} - \frac{16}{3})n \\
&\quad + 5\sqrt{2} + \frac{4\sqrt{77}}{7} - \frac{86\sqrt{35}}{15} + 8\sqrt{11} + \frac{2\sqrt{1190}}{35} + \frac{5\sqrt{14}}{21} - 2\sqrt{5} + 7.
\end{aligned}
$$

(*iii*) **Second Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_2(RHDN3[n]) \\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} M(RHDN3[n]; x, y)|_{x=1}
\end{aligned}
$$

$$
\begin{aligned}
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} \{(6n^2-12n+10)x^4y^4 + 8x^4y^7 + (24n-44)x^4y^{10} \\
&\quad + (12n^2-48n+48)x^4y^{18} + 4x^7y^{10} + 2x^7y^{18} + (4n-10)x^{10}y^{10} \\
&\quad + (8n-20)x^{10}y^{18} + (3n^2-16n+21)x^{18}y^{18}\}|_{x=1} \\
&= S_x^{1/2} J \{\sqrt{4\times 4}\times(6n^2-12n+10)x^4y^4 + \sqrt{4\times 7}\times 8x^4y^7 \\
&\quad + \sqrt{4\times 10}\times(24n-44)x^4y^{10} + \sqrt{4\times 18}\times(12n^2-48n+48)x^4y^{18} \\
&\quad + \sqrt{7\times 10}\times 4x^7y^{10} + \sqrt{7\times 18}\times 2x^7y^{18} \\
&\quad + \sqrt{10\times 10}\times(4n-10)x^{10}y^{10} + \sqrt{10\times 18}\times(8n-20)x^{10}y^{18} \\
&\quad + \sqrt{18\times 18}\times(3n^2-16n+21)x^{18}y^{18}\}|_{x=1} \\
&= S_x^{1/2} \{4\times(6n^2-12n+10)x^8 + \sqrt{28}\times 8x^{11} + \sqrt{40}\times(24n-44)x^{14} \\
&\quad + \sqrt{72}\times(12n^2-48n+48)x^{22} + \sqrt{70}\times 4x^{17} + \sqrt{126}\times 2x^{25} \\
&\quad + 10\times(4n-10)x^{20} + \sqrt{180}\times(8n-20)x^{28} \\
&\quad + 18\times(3n^2-16n+21)x^{36}\}|_{x=1} \\
&= \frac{4}{\sqrt{8}}\times(6n^2-12n+10) + \frac{\sqrt{28}}{\sqrt{11}}\times 8 + \frac{\sqrt{40}}{\sqrt{14}}\times(24n-44) \\
&\quad + \frac{\sqrt{72}}{\sqrt{22}}\times(12n^2-48n+48) + \frac{\sqrt{70}}{\sqrt{17}}\times 4 + \frac{\sqrt{126}}{\sqrt{25}}\times 2 \\
&\quad + \frac{10}{\sqrt{20}}\times(4n-10) + \frac{\sqrt{180}}{\sqrt{28}}\times(8n-20) + \frac{18}{\sqrt{36}}\times(3n^2-16n+21) \\
&= (6\sqrt{2} + \frac{72\sqrt{11}}{11} + 9)n^2 + (-12\sqrt{2} + \frac{48\sqrt{35}}{7} - \frac{288\sqrt{11}}{11} + 4\sqrt{5} \\
&\quad + \frac{24\sqrt{35}}{7} - 48)n + 10\sqrt{2} + \frac{16\sqrt{77}}{11} - \frac{88\sqrt{35}}{7} + \frac{288\sqrt{11}}{11} + \frac{4\sqrt{1190}}{17} \\
&\quad + \frac{6\sqrt{14}}{5} - 10\sqrt{5} - \frac{60\sqrt{35}}{7} + 63.
\end{aligned}
$$

Proof of Theorem 4.6 (The Nirmala Indices for the *SRHDN*3[*n*] Network).

(i) **Nirmala Index:**

$$
\begin{aligned}
&N(SRHDN3[n]) \\
&= D_x^{1/2} J M(SRHDN3[n]; x, y)|_{x=1} \\
&= D_x^{1/2} J \{8(3n^2-6n+4)x^2y^4 + 14x^2y^7 + 40(n-2)x^2y^{10} \\
&\quad + 18(n-2)^2x^2y^{18}\}|_{x=1}
\end{aligned}
$$

$$= D_x^{1/2}\{8(3n^2-6n+4)x^6+14x^9+40(n-2)x^{12}+18(n-2)^2x^{20}\}|_{x=1}$$
$$= \sqrt{6}\times 8(3n^2-6n+4)+\sqrt{9}\times 14+\sqrt{12}\times 40(n-2)$$
$$+\sqrt{20}\times 18(n-2)^2$$
$$= (24\sqrt{6}+36\sqrt{5})n^2+(-48\sqrt{6}+80\sqrt{3}-144\sqrt{5})n+32\sqrt{6}$$
$$+42-160\sqrt{3}+144\sqrt{5}.$$

(*ii*) **First Inverse Nirmala Index:**

$$IN_1(SRHDN3[n])$$
$$= D_x^{1/2}JS_y^{1/2}S_x^{1/2}M(SRHDN3[n];x,y)|_{x=1}$$
$$= D_x^{1/2}JS_y^{1/2}S_x^{1/2}\{8(3n^2-6n+4)x^2y^4+14x^2y^7+40(n-2)x^2y^{10}$$
$$+18(n-2)^2x^2y^{18}\}|_{x=1}$$
$$= D_x^{1/2}J\Big\{\frac{1}{\sqrt{2\times 4}}8(3n^2-6n+4)x^2y^4+\frac{1}{\sqrt{2\times 7}}14x^2y^7$$
$$+\frac{1}{\sqrt{2\times 10}}40(n-2)x^2y^{10}+\frac{1}{\sqrt{2\times 18}}18(n-2)^2x^2y^{18}\Big\}\Big|_{x=1}$$
$$= D_x^{1/2}\Big\{\frac{1}{\sqrt{8}}8(3n^2-6n+4)x^6+\frac{1}{\sqrt{14}}\times 14x^9+\frac{1}{\sqrt{20}}\times 40(n-2)x^{12}$$
$$+\frac{1}{\sqrt{36}}18(n-2)^2x^{20}\Big\}\Big|_{x=1}$$
$$= \frac{\sqrt{6}}{\sqrt{8}}\times 8(3n^2-6n+4)+\frac{\sqrt{9}}{\sqrt{14}}\times 14+\frac{\sqrt{12}}{\sqrt{20}}\times 40(n-2)$$
$$+\frac{\sqrt{20}}{\sqrt{36}}\times 18(n-2)^2$$
$$= (12\sqrt{3}+6\sqrt{5})n^2+(-24\sqrt{3}+8\sqrt{15}-24\sqrt{5})n+16\sqrt{3}+3\sqrt{14}$$
$$-16\sqrt{15}+24\sqrt{5}.$$

(*iii*) **Second Inverse Nirmala Index:**

$$IN_2(SRHDN3[n])$$
$$= S_x^{1/2}JD_y^{1/2}D_x^{1/2}M(SRHDN3[n];x,y)|_{x=1}$$
$$= S_x^{1/2}JD_y^{1/2}D_x^{1/2}\{8(3n^2-6n+4)x^2y^4+14x^2y^7+40(n-2)x^2y^{10}$$
$$+18(n-2)^2x^2y^{18}\}|_{x=1}$$
$$= S_x^{1/2}J\{\sqrt{2\times 4}\times 8(3n^2-6n+4)x^2y^4+\sqrt{2\times 7}\times 14x^2y^7$$

$$
\begin{aligned}
&+\sqrt{2\times 10}\times 40(n-2)x^2y^{10}+\sqrt{2\times 18}\times 18(n-2)^2x^2y^{18}\}\big|_{x=1}\\
&=S_x^{1/2}\{\sqrt{8}\times 8(3n^2-6n+4)x^6+\sqrt{14}\times 14x^9+\sqrt{20}\times 40(n-2)x^{12}\\
&+\sqrt{36}\times 18(n-2)^2x^{20}\}\big|_{x=1}\\
&=\frac{\sqrt{8}}{\sqrt{6}}\times 8(3n^2-6n+4)+\frac{\sqrt{14}}{\sqrt{9}}\times 14+\frac{\sqrt{20}}{\sqrt{12}}\times 40(n-2)\\
&+\frac{\sqrt{36}}{\sqrt{20}}\times 18(n-2)^2\\
&=(16\sqrt{3}+\frac{54\sqrt{5}}{5})n^2+(-32\sqrt{3}+\frac{40\sqrt{15}}{3}-\frac{216\sqrt{5}}{5})n+\frac{64\sqrt{3}}{3}\\
&+\frac{14\sqrt{14}}{3}-\frac{80\sqrt{15}}{3}+\frac{216\sqrt{5}}{5}.
\end{aligned}
$$

Proof of Theorem 4.7 (The Nirmala Indices for the *CHDN*3[*n*] Network).

(*i*) **Nirmala Index:**

$$
\begin{aligned}
&N(CHDN3[n])\\
&=D_x^{1/2}JM(CHDN3[n];x,y)|_{x=1}\\
&=D_x^{1/2}J\{(5n+6)x^4y^4+(6n-4)x^4y^8+(n-2)x^8y^8\}|_{x=1}\\
&=D_x^{1/2}\{(5n+6)x^8+(6n-4)x^{12}+(n-2)x^{16}\}|_{x=1}\\
&=\sqrt{8}\times(5n+6)+\sqrt{12}\times(6n-4)+\sqrt{16}\times(n-2)\\
&=(10\sqrt{2}+12\sqrt{3}+4)n+12\sqrt{2}-8\sqrt{3}-8.
\end{aligned}
$$

(*ii*) **First Inverse Nirmala Index:**

$$
\begin{aligned}
&IN_1(CHDN3[n])\\
&=D_x^{1/2}JS_y^{1/2}S_x^{1/2}M(CHDN3[n];x,y)|_{x=1}\\
&=D_x^{1/2}JS_y^{1/2}S_x^{1/2}\{(5n+6)x^4y^4+(6n-4)x^4y^8+(n-2)x^8y^8\}|_{x=1}\\
&=D_x^{1/2}J\Big\{\frac{1}{\sqrt{4\times 4}}(5n+6)x^4y^4+\frac{1}{\sqrt{4\times 8}}(6n-4)x^4y^8\\
&+\frac{1}{\sqrt{8\times 8}}(n-2)x^8y^8\Big\}\Big|_{x=1}\\
&=D_x^{1/2}\Big\{\frac{1}{4}(5n+6)x^8+\frac{1}{4\sqrt{2}}(6n-4)x^{12}+\frac{1}{8}(n-2)x^{16}\Big\}\Big|_{x=1}
\end{aligned}
$$

$$= \frac{\sqrt{8}}{4} \times (5n+6) + \frac{\sqrt{12}}{4\sqrt{2}} \times (6n-4) + \frac{\sqrt{16}}{8} \times (n-2)$$
$$= (\frac{5\sqrt{2}}{2} + \frac{3\sqrt{6}}{2} + \frac{1}{2})n + 3\sqrt{2} - \sqrt{6} - 1.$$

(*iii*) **Second Inverse Nirmala Index:**

$$\begin{aligned}
&IN_2(CHDN3[n]) \\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} M(CHDN3[n]; x, y)|_{x=1} \\
&= S_x^{1/2} J D_y^{1/2} D_x^{1/2} \{(5n+6)x^4y^4 + (6n-4)x^4y^8 + (n-2)x^8y^8\}|_{x=1} \\
&= S_x^{1/2} J \{\sqrt{4\times 4} \times (5n+6)x^4y^4 + \sqrt{4\times 8} \times (6n-4)x^4y^8 \\
&\quad + \sqrt{8\times 8} \times (n-2)x^8y^8\}\big|_{x=1} \\
&= S_x^{1/2} \{4 \times (5n+6)x^8 + 4\sqrt{2} \times (6n-4)x^{12} + 8 \times (n-2)x^{16}\}\big|_{x=1} \\
&= \frac{4}{\sqrt{8}} \times (5n+6) + \frac{4\sqrt{2}}{\sqrt{12}}(6n-4) + \frac{8}{\sqrt{16}} \times (n-2) \\
&= (5\sqrt{2} + 4\sqrt{6} + 2)n + 6\sqrt{2} - \frac{8\sqrt{6}}{3} - 4.
\end{aligned}$$

Proof of Theorem 4.8 (The Nirmala Indices for the *SCHDN*3[*n*] Network).

(*i*) **Nirmala Index:**

$$\begin{aligned}
&N(SCHDN3[n]) \\
&= D_x^{1/2} J M(SCHDN3[n]; x, y)|_{x=1} \\
&= D_x^{1/2} J\{(16n+8)x^2y^4 + (8n-8)x^2y^8\}|_{x=1} \\
&= D_x^{1/2} \{(16n+8)x^6 + (8n-8)x^{10}\}|_{x=1} \\
&= \sqrt{6} \times (16n+8) + \sqrt{10} \times (8n-8) \\
&= (16\sqrt{6} + 8\sqrt{10})n + 8\sqrt{6} - 8\sqrt{10}.
\end{aligned}$$

(*ii*) **First Inverse Nirmala Index:**

$$\begin{aligned}
&IN_1(SCHDN3[n]) \\
&= D_x^{1/2} J S_y^{1/2} S_x^{1/2} M(SCHDN3[n]; x, y)|_{x=1} \\
&= D_x^{1/2} J S_y^{1/2} S_x^{1/2} \{(16n+8)x^2y^4 + (8n-8)x^2y^8\}|_{x=1}
\end{aligned}$$

$$= D_x^{1/2} J\Big\{\frac{1}{\sqrt{2\times 4}}(16n+8)x^2y^4 + \frac{1}{\sqrt{2\times 8}}(8n-8)x^2y^8\Big\}\Big|_{x=1}$$
$$= D_x^{1/2}\Big\{\frac{1}{\sqrt{8}}(16n+8)x^6 + \frac{1}{4}(8n-8)x^{10}\Big\}\Big|_{x=1}$$
$$= \frac{\sqrt{6}}{\sqrt{8}}\times(16n+8) + \frac{\sqrt{10}}{4}\times(8n-8)$$
$$= (8\sqrt{3}+2\sqrt{10})n + 4\sqrt{3} - 2\sqrt{10}.$$

(*iii*) **Second Inverse Nirmala Index:**

$$IN_2(SCHDN3[n])$$
$$= S_x^{1/2} J D_y^{1/2} D_x^{1/2} M(SCHDN3[n]; x, y)|_{x=1}$$
$$= S_x^{1/2} J D_y^{1/2} D_x^{1/2}\{(16n+8)x^2y^4 + (8n-8)x^2y^8\}|_{x=1}$$
$$= S_x^{1/2} J\{\sqrt{2\times 4}\times(16n+8)x^2y^4 + \sqrt{2\times 8}\times(8n-8)x^2y^8\}\big|_{x=1}$$
$$= S_x^{1/2}\{\sqrt{8}\times(16n+8)x^6 + 4\times(8n-8)x^{10}\}\big|_{x=1}$$
$$= \frac{\sqrt{8}}{\sqrt{6}}\times(16n+8) + \frac{4}{\sqrt{10}}(8n-8)$$
$$= (\frac{32\sqrt{3}}{3} + \frac{16\sqrt{10}}{5})n + \frac{16\sqrt{3}}{3} - \frac{16\sqrt{10}}{5}.$$

□

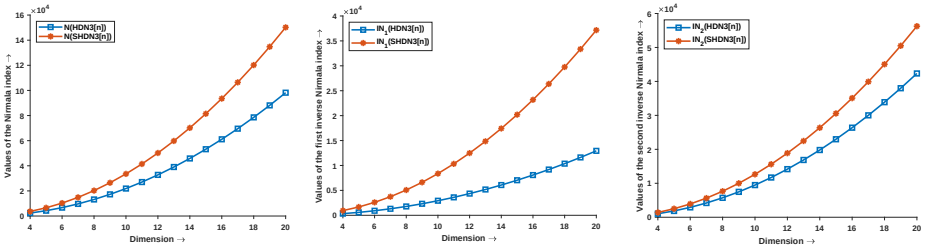
5 Numerical and Graphical Demonstration of the Nirmala Indices for the Hex-derived Networks and Their Subdivisions

This segment deals with some comparative assessments of the Nirmala indices for the hex-derived networks of type three and their subdivisions.

Table 10: Computation of Nirmala indices of the hex-derived and subdivided hex-derived networks of type three at different values of n.

Nirmala Indices / Dimension	$N(HDN3[n])$	$N(SHDN3[n])$	$N(THDN3[n])$	$N(STHDN3[n])$	$N(RHDN3[n])$	$N(SRHDN3[n])$	$N(CHDN3[n])$	$N(SCHDN3[n])$	$IN_1(HDN3[n])$	$IN_1(SHDN3[n])$	$IN_1(THDN3[n])$	$IN_1(STHDN3[n])$	$IN_1(RHDN3[n])$	$IN_1(SRHDN3[n])$	$IN_1(CHDN3[n])$	$IN_1(SCHDN3[n])$	$IN_2(HDN3[n])$	$IN_2(SHDN3[n])$	$IN_2(THDN3[n])$	$IN_2(STHDN3[n])$	$IN_2(RHDN3[n])$	$IN_2(SRHDN3[n])$	$IN_2(CHDN3[n])$	$IN_2(SCHDN3[n])$
n=2	–	–	–	–	–	–	72.9677	123.2778	–	–	–	–	–	–	16.2127	40.9656	–	–	–	–	–	–	35.6914	56.3073
n=3	–	1602.7883	–	–	–	515.8095	111.8944	187.7679	–	423.4809	–	–	–	145.6919	23.9225	61.1465	–	631.4434	–	–	–	213.3392	54.5604	84.9018
n=4	2361.9245	3658.4320	365.3545	582.2833	757.1112	1189.8076	150.8211	252.2579	332.9569	942.7227	59.8650	163.8153	115.2680	320.8480	31.6322	81.3275	1039.6510	1414.3896	168.00001	240.1733	340.9680	475.9917	73.4294	113.4963
n=5	4244.4666	6549.7930	667.8367	1052.9604	1373.3504	2142.3782	189.7479	316.7480	586.1649	1667.1707	103.3378	286.6321	200.9421	564.4062	39.3420	101.5085	1857.2936	2508.5100	302.4627	424.1970	611.7045	842.3688	92.2984	142.0908
n=6	6674.5420	10276.8712	1061.5745	1662.9237	2172.1007	3373.5212	228.6746	381.2380	910.6282	2596.8247	158.6864	443.6500	310.3679	876.3664	47.0518	121.6894	2910.1009	3913.8044	476.1194	660.0831	960.8291	1312.4707	111.1675	170.6853
n=7	9652.1508	14839.6666	1546.5678	2412.1732	3153.3622	4883.2366	267.6014	445.7281	1306.3469	3731.6849	225.9110	634.8689	443.5456	1256.7287	54.7615	141.8704	4198.0728	5630.2729	688.9703	947.8316	1388.3419	1886.2972	130.0365	199.2798
n=8	13177.2929	20238.1792	2122.8167	3300.7089	4317.1347	6671.5243	306.52812	510.2182	1773.3209	5071.7511	305.0114	860.2888	600.4750	1705.4929	62.4713	162.0513	5721.2092	7657.9155	941.0152	1287.4423	1894.2429	2563.8484	148.9055	227.8743
n=9	17249.9682	26472.4090	2790.3211	4328.5307	5663.4184	8738.3845	345.4549	574.7082	2311.5502	6617.0235	395.9877	1119.9097	781.1562	2222.6593	70.1811	182.2323	7479.5102	9996.7322	1232.2543	1678.9155	2478.5322	3345.1243	167.7746	256.4688
n=10	21870.1770	33542.3560	3549.0810	5495.6388	7192.2131	11083.8171	384.3816	639.1983	2921.0349	8367.5020	498.8399	1413.7316	985.5892	2808.2276	77.8908	202.4133	9472.9759	12646.7230	1562.6874	2122.2510	3141.2096	4230.1249	186.6436	285.0633
n=11	27037.9189	41448.0202	4399.0966	6802.0331	8903.5189	13707.8221	423.3084	703.6883	3601.7749	10323.1866	613.5680	1741.7546	1213.7740	3462.1980	85.6006	222.5942	11701.6061	15607.8878	1932.3146	2617.4488	3882.2752	5218.8502	205.5126	313.6578
n=12	32753.1943	50189.4016	5340.3676	8247.7136	10797.3359	16610.3995	462.2351	768.1784	4353.7703	12484.0772	740.1720	2103.9786	1465.7105	4184.5704	93.3104	242.7752	14165.4009	18880.2268	2341.1360	3164.5090	4701.7290	6311.3002	224.3816	342.2523
n=13	39016.0029	59766.5003	6372.8942	9832.6803	12873.6640	19791.5493	501.1619	832.6684	5177.0210	14850.1740	878.6519	2500.4035	1741.3989	4975.3449	101.0201	262.9562	16864.3603	22463.7398	2789.1514	3763.4315	5599.5710	7507.4749	243.2507	370.8468
n=14	45826.3448	70179.3161	7496.6764	11556.9332	15132.5031	23251.2715	540.0886	897.1585	6071.5271	17421.4769	1029.0077	2931.0295	2040.8390	5834.5214	108.7299	283.1371	19798.4842	26358.4269	3276.3609	4414.2164	6575.8012	8807.3743	262.1197	399.4413
n=15	53184.2201	81427.8491	8711.7141	13420.4723	17573.8534	26989.5661	579.0153	961.6486	7037.2885	20197.9859	1191.2393	3395.8566	2364.0309	6762.0999	116.4397	303.3181	22967.7728	30564.2881	3802.7646	5116.8636	7630.4196	10210.9984	280.9887	428.0358
n=16	61089.6286	93512.0994	10018.0074	15423.2976	20197.7148	31006.4331	617.9421	1026.1386	8074.3052	23179.7010	1365.3469	3894.8846	2710.9746	7758.0804	124.1495	323.4990	26372.2260	35081.3233	4368.3623	5871.3731	8763.4262	11718.3472	299.8577	456.6303
n=17	69542.5705	106432.0668	11415.5562	17565.4091	23004.0872	35301.8725	656.8688	1090.6287	9182.5773	26366.6222	1551.3303	4428.1137	3081.6700	8822.4630	131.8592	343.6799	30011.8437	39909.5327	4973.1541	6677.7451	9974.8210	13329.4206	318.7268	485.2248
n=18	78543.0457	120187.7515	12904.3606	19846.8068	25992.9708	39875.8843	695.7956	1155.1187	10362.1047	29758.7496	1749.1897	4995.5438	3476.1172	9955.2477	139.5690	363.8610	33886.6261	45048.9161	5617.1401	7535.9793	11264.6039	15044.2188	337.5958	513.8193
n=19	88091.0542	134779.1533	14484.4205	22267.4906	29164.3655	44728.4685	734.7223	1219.6088	11612.8875	33356.0830	1958.9249	5597.1749	3894.3163	11156.4343	147.2788	384.0419	37996.5730	50499.4736	6300.3201	8446.0759	12632.7751	16862.7417	356.4648	542.4138
n=20	98186.5960	150206.2724	16155.7360	24827.4607	32518.2713	49859.6251	773.6491	1284.0988	12934.9256	37158.6225	2180.5360	6233.0070	4336.2671	12426.0230	154.9885	404.2229	42341.6845	56261.2052	7022.6942	9408.0349	14079.3345	18784.9892	375.3338	571.0083

The values of the Nirmala indices for the networks and their subdivisions are reported in Table 10 for various dimensions n (where, $2 \leq n \leq 20$) and have pictorially illustrated the determined results of the table in Figures 5 to 15 with the help of MATLAB R2019a software to obtain relations among them.

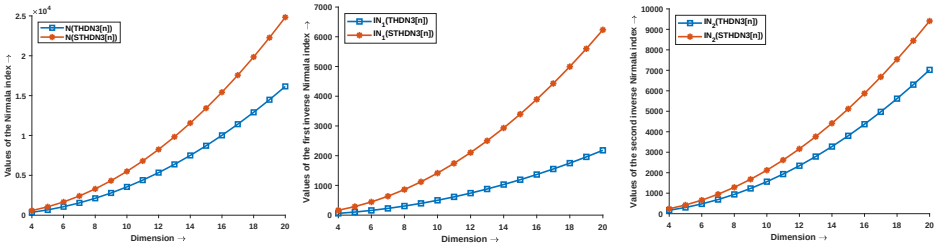


(a) Comparison of $N(HDN3[n])$ with $N(SHDN3[n])$.

(b) Comparison of $IN_1(HDN3[n])$ with $IN_1(SHDN3[n])$.

(c) Comparison of $IN_2(HDN3[n])$ with $IN_2(SHDN3[n])$.

Figure 5: A comparative analysis of the Nirmala indices between the $HDN3[n]$ and $SHDN3[n]$ networks, where $4 \leq n \leq 20$.



(a) Comparison of $N(THDN3[n])$ with $N(STHDN3[n])$.

(b) Comparison of $IN_1(THDN3[n])$ with $IN_1(STHDN3[n])$.

(c) Comparison of $IN_2(THDN3[n])$ with $IN_2(STHDN3[n])$.

Figure 6: A comparative analysis of the Nirmala indices between the $THDN3[n]$ and $STHDN3[n]$ networks, where $4 \leq n \leq 20$.

In the depiction of the plots, we have taken into account the three viewpoints and obtained the following outcomes.

(I) In the first approach, we have drawn a comparative plot for each of the Nirmala indices individually between a hex-derived network and its corresponding subdivision for all our considered hex-derived networks in Figures 5 to 8, and hence we conclude the followings:

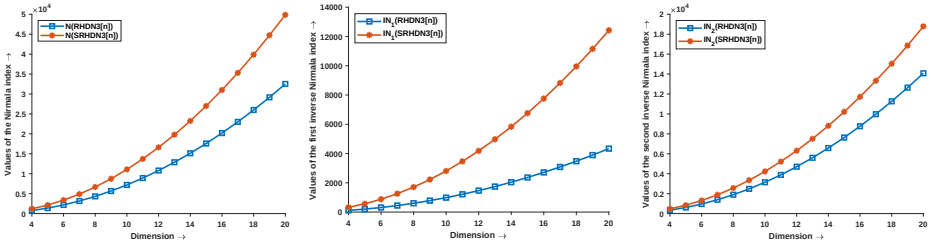
(i) **for the hex-derived network of third type:** $N(HDN3[n]) \leq N(SHDN3[n])$,

$IN_1(HDN3[n]) \leq IN_1(SHDN3[n])$ and
$IN_2(HDN3[n]) \leq IN_2(SHDN3[n])$, where $4 \leq n \leq 20$,

(ii) **for the triangular hex-derived network of third type:**
$N(THDN3[n]) \leq N(STHDN3[n])$,
$IN_1(THDN3[n]) \leq IN_1(STHDN3[n])$ and
$IN_2(THDN3[n]) \leq IN_2(STHDN3[n])$, where $4 \leq n \leq 20$,

(iii) **for the rectangular hex-derived network of third type:**
$N(RHDN3[n]) \leq N(SRHDN3[n])$,
$IN_1(RHDN3[n]) \leq IN_1(SRHDN3[n])$ and
$IN_2(RHDN3[n]) \leq IN_2(SRHDN3[n])$, where $4 \leq n \leq 20$,

(iv) **for the chain hex-derived network of third type:**
$N(CHDN3[n]) \leq N(SCHDN3[n])$,
$IN_1(CHDN3[n]) \leq IN_1(SCHDN3[n])$ and
$IN_2(CHDN3[n]) \leq IN_2(SCHDN3[n])$, where $2 \leq n \leq 20$.

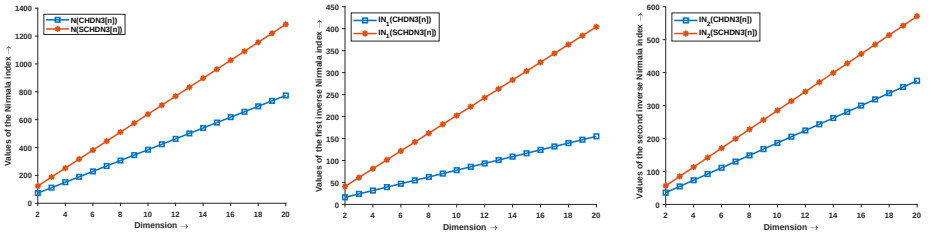


(*a*) Comparison of $N(RHDN3[n])$ with $N(SRHDN3[n])$.

(*b*) Comparison of $IN_1(RHDN3[n])$ with $IN_1(SRHDN3[n])$.

(*c*) Comparison of $IN_2(RHDN3[n])$ with $IN_2(SRHDN3[n])$.

Figure 7: A comparative analysis of the Nirmala indices between the $RHDN3[n]$ and $SRHDN3[n]$ networks, where $4 \leq n \leq 20$.

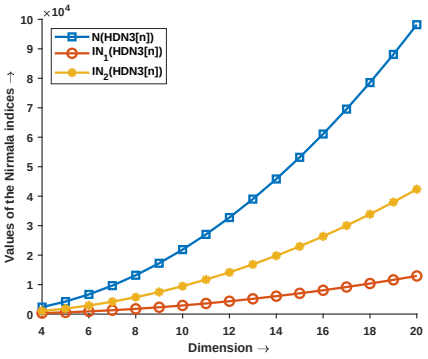


(*a*) Comparison of $N(CHDN3[n])$ with $N(SCHDN3[n])$.

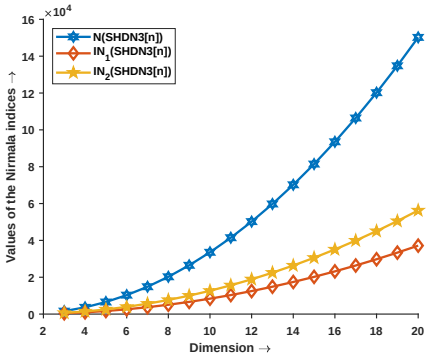
(*b*) Comparison of $IN_1(CHDN3[n])$ with $IN_1(SCHDN3[n])$.

(*c*) Comparison of $IN_2(CHDN3[n])$ with $IN_2(SCHDN3[n])$.

Figure 8: A comparative analysis of the Nirmala indices between the $CHDN3[n]$ and $SCHDN3[n]$ networks, where $2 \leq n \leq 20$.

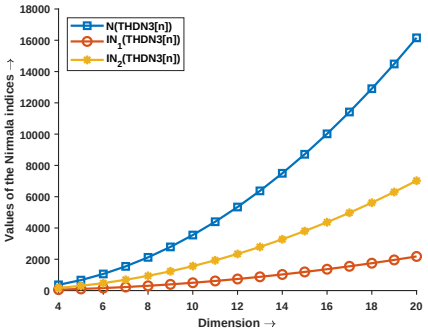


(a) Portray of the Nirmala indices of the $HDN3[n]$ network for $4 \leq n \leq 20$.

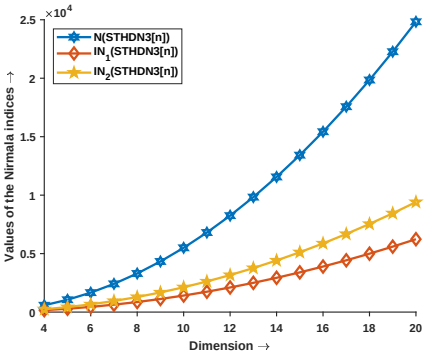


(b) Portray of the Nirmala indices of the $SHDN3[n]$ network for $3 \leq n \leq 20$.

Figure 9: Plot of the Nirmala indices for the $HDN3[n]$ and $SHDN3[n]$ networks.



(a) Portray of the Nirmala indices of the $THDN3[n]$ network for $4 \leq n \leq 20$.



(b) Portray of the Nirmala indices of the $STHDN3[n]$ network for $4 \leq n \leq 20$.

Figure 10: Plot of the Nirmala indices for the $THDN3[n]$ and $STHDN3[n]$ networks.

(II) In the second strategy, we plot all the Nirmala indices in a single frame by considering each of the hex-derived and their subdivided networks, one by one in Figures 9 to 12, and the outcomes are as follows:

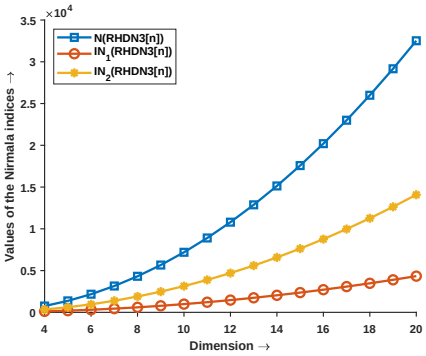
(i) **for the hex-derived and subdivided hex-derived networks of third type:**
$IN_1(HDN3[n]) \leq IN_2(HDN3[n]) \leq N(HDN3[n])$,
where $4 \leq n \leq 20$ and

$IN_1(SHDN3[n]) \leq IN_2(SHDN3[n]) \leq N(SHDN3[n])$,
where $3 \leq n \leq 20$,

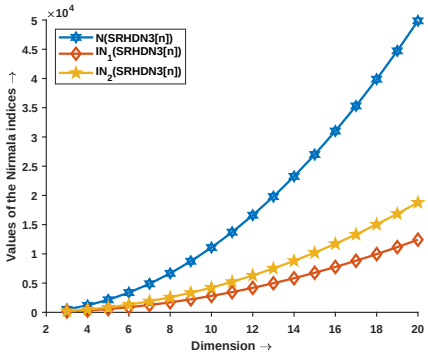
(ii) **for the triangular hex-derived and subdivided triangular hex-derived networks of third type:**
$IN_1(THDN3[n]) \leq IN_2(THDN3[n]) \leq N(THDN3[n])$,
where $4 \leq n \leq 20$ and
$IN_1(STHDN3[n]) \leq IN_2(STHDN3[n]) \leq N(STHDN3[n])$,
where $4 \leq n \leq 20$,

(iii) **for the rectangular hex-derived and subdivided rectangular hex-derived networks of third type:**
$IN_1(RHDN3[n]) \leq IN_2(RHDN3[n]) \leq N(RHDN3[n])$,
where $4 \leq n \leq 20$ and
$IN_1(SRHDN3[n]) \leq IN_2(SRHDN3[n]) \leq N(SRHDN3[n])$,
where $3 \leq n \leq 20$,

(iv) **for the chain hex-derived and subdivided chain hex-derived networks of third type:**
$IN_1(CHDN3[n]) \leq IN_2(CHDN3[n]) \leq N(CHDN3[n])$,
where $2 \leq n \leq 20$ and
$IN_1(SCHDN3[n]) \leq IN_2(SCHDN3[n]) \leq N(SCHDN3[n])$,
where $2 \leq n \leq 20$.

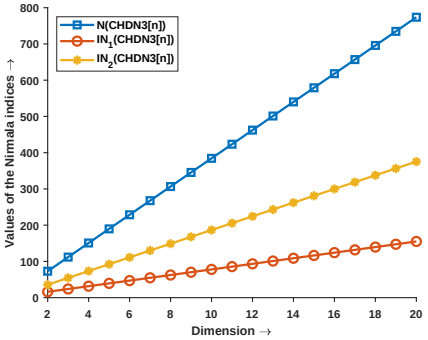


(a) Portray of the Nirmala indices of the $RHDN3[n]$ network for $4 \leq n \leq 20$.

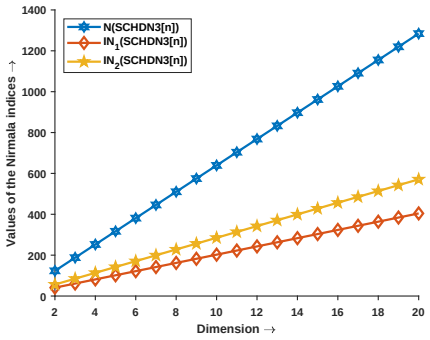


(b) Portray of the Nirmala indices of the $SRHDN3[n]$ network for $3 \leq n \leq 20$.

Figure 11: Plot of the Nirmala indices for the $RHDN3[n]$ and $SRHDN3[n]$ networks.

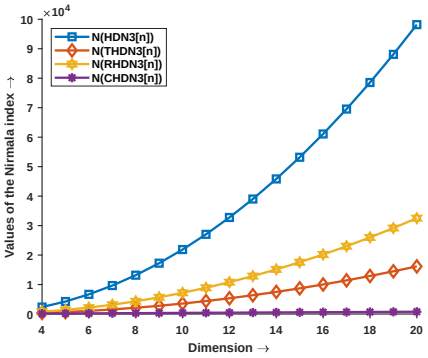


(*a*) Portray of the Nirmala indices of the $CHDN3[n]$ network for $2 \leq n \leq 20$.

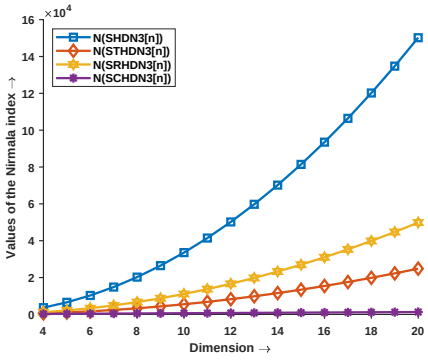


(*b*) Portray of the Nirmala indices of the $SCHDN3[n]$ network for $2 \leq n \leq 20$.

Figure 12: Plot of the Nirmala indices for the $CHDN3[n]$ and $SCHDN3[n]$ networks.



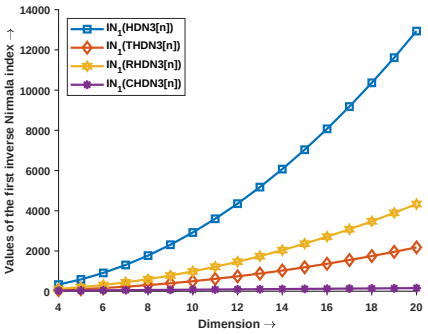
(*a*) Comparative plot of the Nirmala index for the hex-derived networks of type three.



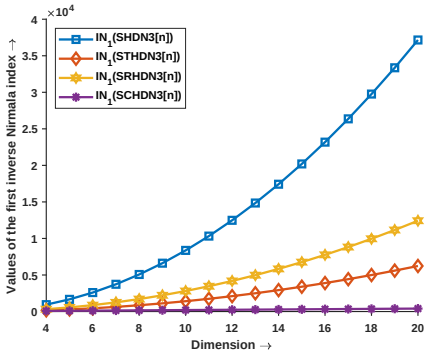
(*b*) Comparative plot of the Nirmala index for the subdivided hex-derived networks of type three.

Figure 13: Graphical illustration of the Nirmala index for the hex-derived and subdivided hex-derived networks of third type, where $4 \leq n \leq 20$.

(III) In the last and final approach, we portray the Nirmala index for all considered hex-derived networks of type three in one frame and subdivided hex-derived networks of type three in another frame. We perform the same for the first inverse Nirmala index and the second inverse Nirmala index, respectively. See Figures 13 to 15, respectively and we have the following observations:

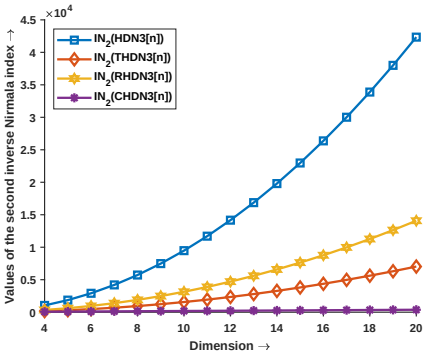


(a) Comparative plot of the first inverse Nirmala index for the hex-derived networks of type three.

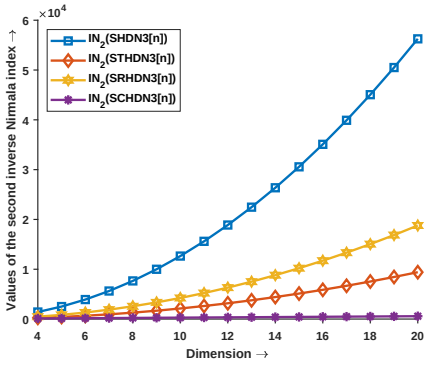


(b) Comparative plot of the first inverse Nirmala index for the subdivided hex-derived networks of type three.

Figure 14: Graphical illustration of the first inverse Nirmala index for the hex-derived and subdivided hex-derived networks of third type, where $4 \leq n \leq 20$.



(a) Comparative plot of the second inverse Nirmala index for the hex-derived networks of type three.



(b) Comparative plot of the second inverse Nirmala index for the subdivided hex-derived networks of type three.

Figure 15: Graphical illustration of the second inverse Nirmala index for the hex-derived and subdivided hex-derived networks of third type, where $4 \leq n \leq 20$.

(i) **Comparison of the Nirmala index:**
$N(CHDN3[n]) \leq N(THDN3[n]) \leq N(RHDN3[n])$
$\leq N(HDN3[n])$, where $4 \leq n \leq 20$ and
$N(SCHDN3[n]) \leq N(STHDN3[n]) \leq N(SRHDN3[n])$
$\leq N(SHDN3[n])$, where $4 \leq n \leq 20$,

(ii) **Comparison of the first inverse Nirmala index:**
$IN_1(CHDN3[n]) \leq IN_1(THDN3[n]) \leq IN_1(RHDN3[n])$
$\leq IN_1(HDN3[n])$, where $4 \leq n \leq 20$ and
$IN_1(SCHDN3[n]) \leq IN_1(STHDN3[n]) \leq IN_1(SRHDN3[n])$
$\leq IN_1(SHDN3[n])$, where $4 \leq n \leq 20$,

(iii) **Comparison of the second inverse Nirmala index:**
$IN_2(CHDN3[n]) \leq IN_2(THDN3[n]) \leq IN_2(RHDN3[n])$
$\leq IN_2(HDN3[n])$, where $4 \leq n \leq 20$ and
$IN_2(SCHDN3[n]) \leq IN_2(STHDN3[n]) \leq IN_2(SRHDN3[n])$
$\leq IN_2(SHDN3[n])$, where $4 \leq n \leq 20$.

6 Conclusions

In this article, we have considered different hex-derived networks of type three: $HDN3[n]$, $THDN3[n]$, $RHDN3[n]$, $CHDN3[n]$, and their subdivisions $SHDN3[n]$, $STHDN3[n]$, $SRHDN3[n]$, $SCHDN3[n]$ networks, respectively. We first calculated the Nirmala indices for the studied hex-derived networks and their associated subdivisions by their standard degree-dependent mathematical formulas. Afterward, by recollecting the M-polynomial expressions of the concede networks, we evaluated the Nirmala indices using their M-polynomial based derivation formulas. It is evident that the M-polynomial formulas approach to determine the Nirmala indices make the process simple than calculating them using the degree-dependent mathematical formulas.

Further, we can observe that the graph of the Nirmala indices for the $HDN3[n]$, $SHDN3[n]$, $THDN3[n]$, $STHDN3[n]$, $RHDN3[n]$ and $SRHDN3[n]$ networks have a curved shape because the expression of the Nirmala indices for these structures is in quadratic form. Whereas, the graph of the Nirmala indices for the $CHDN3[n]$ and $SCHDN3[n]$ networks is in the form of a straight line, i.e., they increase linearly because the Nirmala indices for both of these structures have linear expressions. These structural variations arise due to the dependency of the indices on the vertices and edges of the structure.

At last, we can conclude that the values of the Nirmala indices for the hex-derived networks are always lesser than those of their concerned subdivided hex-derived networks. Additionally, when we plotted all the Nirmala indices for each of the individual hex-derived and subdivided hex-derived networks, we found the relation $IN_1 \leq IN_2 \leq N$. Finally, another intriguing result we obtained here is that when we depicted each of the

Nirmala indices for all considered hex-derived networks of type three in one frame and subdivided hex-derived networks of type three in another frame, the values of the each of the Nirmala indices are in the order: $CHDN3[n] \leq THDN3[n] \leq RHDN3[n] \leq HDN3[n]$ for the hex-derived networks, and $SCHDN3[n] \leq STHDN3[n] \leq SRHDN3[n] \leq SHDN3[n]$ for the subdivided hex-derived networks.

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References

[1] Alexandru T. Balaban. *Chemical applications of graph theory*. Academic Press, 1976.

[2] M.-S. Chen, Kang G. Shin, and Dilip D. Kandlur. Addressing, routing, and broadcasting in hexagonal mesh multiprocessors. *IEEE Transactions on Computers*, 39(1):10–18, 1990. doi:10.1109/12.46277.

[3] Shibsankar Das and Virendra Kumar. Investigation of closed derivation formula for GQ and QG indices of a graph via M-polynomial. *Iranian Journal of Mathematical Chemistry*, 13(2):129–144, 2022. doi:10.22052/ijmc.2022.246172.1614.

[4] Shibsankar Das and Virendra Kumar. On M-polynomial of the two-dimensional Silicon-Carbons. *Palestine Journal of Mathematics*, 11(Special Issue II):136–157, 2022.

[5] Shibsankar Das and Shikha Rai. M-polynomial and related degree-based topological indices of the third type of hex-derived network. *Nanosystems: Physics, Chemistry, Mathematics*, 11(3):267–274, 2020. doi:10.17586/2220-8054-2020-11-3-267-274.

[6] Shibsankar Das and Shikha Rai. M-polynomial and related degree-based topological indices of the third type of chain hex-derived network. *Malaya Journal of Matematik (MJM)*, 8(4):1842–1850, 2020. doi:10.26637/MJM0804/0085.

[7] Shibsankar Das and Shikha Rai. Topological characterization of the third type of triangular hex-derived networks. *Scientific Annals of Computer Science*, 31(2):145–161, 2021. doi:10.7561/SACS.2021.2.145.

[8] Shibsankar Das and Shikha Rai. Computational analysis of topological indices for hexderived networks of type 3 and their subdivisions. *Grenze International Journal of Engineering and Technology*, 8(2):1162–1169, 2022.

[9] Shibsankar Das and Shikha Rai. Degree-based topological descriptors of type 3 rectangular hex-derived networks. *Bulletin of the Institute of Combinatorics and its Applications*, 95:21–37, 2022.

[10] Shibsankar Das and Shikha Rai. On M-polynomial and associated topological descriptors of subdivided hex-derived network of type three. *Computational Technologies*, 27(4):84–97, 2022. doi:10.25743/ICT.2022.27.4.007.

[11] Shibsankar Das and Shikha Rai. On closed derivation formulas of Nirmala indices from the M-polynomial of a graph. *Journal of the Indian Chemical Society*, 100(6):101017, 2023. doi:10.1016/j.jics.2023.101017.

[12] Shibsankar Das, Shikha Rai, and Virendra Kumar. On topological indices of Molnupiravir and its QSPR modelling with some other antiviral drugs to treat COVID-19 patients. *Journal of Mathematical Chemistry*, 2023 (Article in Press.). doi:10.1007/s10910-023-01518-z.

[13] Shibsankar Das, Shikha Rai, and Modhuleena Mandal. M-polynomial and correlated topological indices of antiviral drug Molnupiravir used as a therapy for COVID-19. *Polycyclic Aromatic Compounds*, 43(8):7027–7041, 2023. doi:10.1080/10406638.2022.2131854.

[14] Hanyuan Deng, Jianguang Yang, and Fangli Xia. A general modeling of some vertex-degree based topological indices in benzenoid systems and phenylenes. *Computers & Mathematics with Applications*, 61(10):3017–3023, 2011. doi:10.1016/j.camwa.2011.03.089.

[15] Emeric Deutsch and Sandi Klavžar. M-polynomial and degree-based topological indices. *Iranian Journal of Mathematical Chemistry*, 6(2):93–102, 2015. doi:10.22052/ijmc.2015.10106.

[16] Edward J. Farrell. An introduction to matching polynomials. *Journal of Combinatorial Theory, Series B*, 27(1):75–86, 1979. doi:10.1016/0095-8956(79)90070-4.

[17] Haruo Hosoya. On some counting polynomials in chemistry. *Discrete Applied Mathematics*, 19(1-3):239–257, 1988. doi:10.1016/0166-218X(88)90017-0.

[18] Zahid Iqbal, Adnan Aslam, Muhammad Ishaq, and Wei Gao. On computations of topological descriptors of kagome lattice. *Polycyclic Aromatic Compounds*, 42(8):4895–4909, 2022. doi:10.1080/10406638.2021.1923537.

[19] K. Julietraja, P. Venugopal, S. Prabhu, and Jia-Bao Liu. M-polynomial and degree-based molecular descriptors of certain classes of benzenoid systems. *Polycyclic Aromatic Compounds*, 42(6):3450–3477, 2022. doi:10.1080/10406638.2020.1867205.

[20] Shin Min Kang, Waqas Nazeer, Manzoor Ahmad Zahid, Abdul Rauf Nizami, Adnan Aslam, and Mobeen Munir. M-polynomials and topological indices of hex-derived networks. *Open Physics*, 16(1):394–403, 2018. doi:10.1515/phys-2018-0054.

[21] V. R. Kulli. Nirmala index. *International Journal of Mathematics Trends and Technology*, 67(3):8–12, 2021. doi:10.14445/22315373/IJMTT-V67I3P502.

[22] V. R. Kulli, V. Lokesha, and K. Nirupadi. Computation of inverse Nirmala indices of certain nanostructures. *International Journal of Mathematical Combinatorics*, 2:33–40, 2021.

[23] Young Chel Kwun, Mobeen Munir, Waqas Nazeer, Shazia Rafique, and Shin Min Kang. M-polynomials and topological indices of V-phenylenic nanotubes and nanotori. *Scientific Reports*, 7(1):1–9, 2017. doi:10.1038/s41598-017-08309-y.

[24] Paul Manuel, Rajan Bharati, Indra Rajasingh, and Chris Monica M. On minimum metric dimension of honeycomb networks. *Journal of Discrete Algorithms*, 6(1):20–27, 2008. doi:10.1016/j.jda.2006.09.002.

[25] Sourav Mondal, Muhammad Kamran Siddiqui, Nilanjan De, and Anita Pal. Neighborhood M-polynomial of crystallographic structures.

Biointerface Research in Applied Chemistry, 11(2):9372–9381, 2020. doi:10.33263/BRIAC112.93729381.

[26] Mobeen Munir, Waqas Nazeer, Shazia Rafique, and Shin Min Kang. M-polynomial and degree-based topological indices of polyhex nanotubes. *Symmetry*, 8(12):149, 2016. doi:10.3390/sym8120149.

[27] F. Garcia Nocetti, Ivan Stojmenovic, and Jingyuan Zhang. Addressing and routing in hexagonal networks with applications for tracking mobile users and connection rerouting in cellular networks. *IEEE Transactions on Parallel and Distributed Systems*, 13(9):963–971, 2002. doi:10.1109/TPDS.2002.1036069.

[28] Md. Saidur Rahman. *Basic graph theory*. Springer Publishing Company, Incorporated, 1st edition, 2017.

[29] Shikha Rai and Shibsankar Das. M-polynomial and degree-based topological indices of subdivided chain hex-derived network of type 3. In Isaac Woungang, Sanjay Kumar Dhurandher, Kiran Kumar Pattanaik, Anshul Verma, and Pradeepika Verma, editors, *Advanced Network Technologies and Intelligent Computing*, volume 1534 of *Communications in Computer and Information Science (CCIS) series*, pages 410–424. Springer International Publishing, Cham, 2022. doi:10.1007/978-3-030-96040-7_33.

[30] Shikha Rai and Shibsankar Das. Comparative analysis of M-polynomial based topological indices between poly hex-derived networks and their subdivisions. *Proyecciones (Antofagasta, On line)*, Accepted on October-2023 (Article in Press).

[31] F Simon Raj and A George. Network embedding on Planar Octahedron networks. In *2015 IEEE International Conference on Electrical, Computer and Communication Technologies (ICECCT)*, pages 1–6. IEEE, 2015. doi:10.1109/ICECCT.2015.7226174.

[32] Nenad Trinajstić. *Chemical graph theory*. CRC Press, 2nd edition, 1992.